

# Online Appendix for: Term Structure, Forecast Revision and the Signaling Channel of Monetary Policy

Donghai Zhang\*  
University of Bonn

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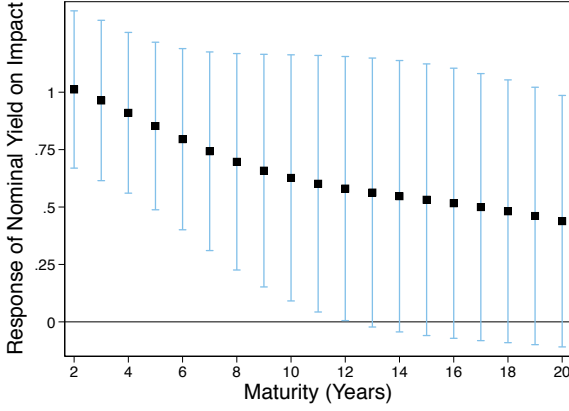
The appendices are organized as follows. In Appendix A, I show that the one-step approach used in the empirical section of the paper produces similar results as the two-step approach. In Appendix B, I provide robustness checks for the empirical results presented in the paper. Appendix C presents the solution of the model. In Appendix D, I describe the measure equations that are used to estimate the model and I conduct posterior convergence checks.

## A The One-step Approach v.s the Two-step Approach

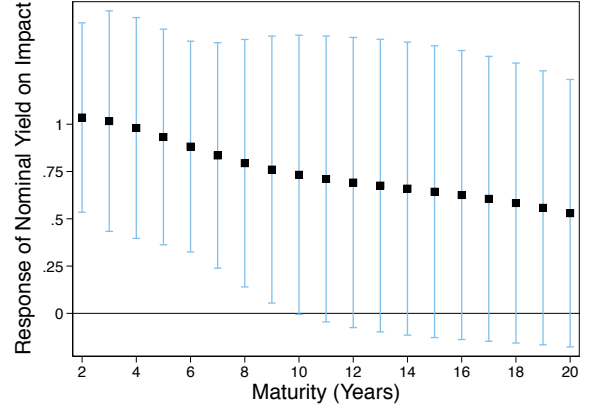
In the paper, I discuss that the one-step approach that is used to produce Figure 1 and Figure 6, which can be found in Panel a) and Panel c) in Figure (1) of the current appendix, is equivalent to the two-step approach: first, clean those HFI monetary surprises using Greenbook forecasts; and second, use those cleaned monetary surprises as instruments. The results using the two-step approach are reported in Panels b) and d) in Figure (1). It is evident that the two approaches produce nearly the same results. The one-step approach employed in the paper has the advantage that the standard errors are free of construction error.

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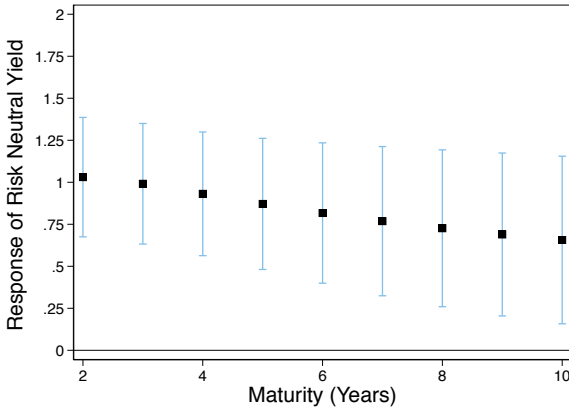
\*Email: [donghai.zhang@uni-bonn.de](mailto:donghai.zhang@uni-bonn.de)



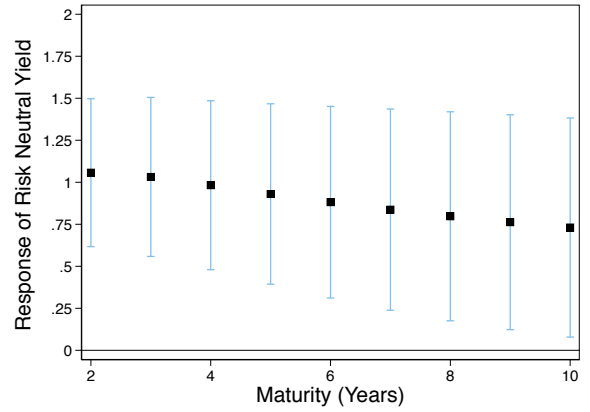
(a) Figure 1 in the Paper



(b) Figure 1 based on Two-step Approach



(c) Figure 6 in the Paper



(d) Figure 6 based on Two-step Approach

**Figure 1:** Panel a) and c) report Figure 1 and Figure 6 in the paper, respectively. Panel b) and d) report the alternative Figures based on the two-step approach.

## B Additional Empirical Results: Robustness Checks

**Monetary Shocks and the Yield Curve** This section shows that the empirical evidences presented in section (2) of the paper are robust to: i) the use of an alternative instrument set; ii) pre-2008 sample; iii) the use of daily changes in nominal yields instead of a two-day window; iv) excluding Greenbook forecasts; and v) the use of inflation indexed interest rate.

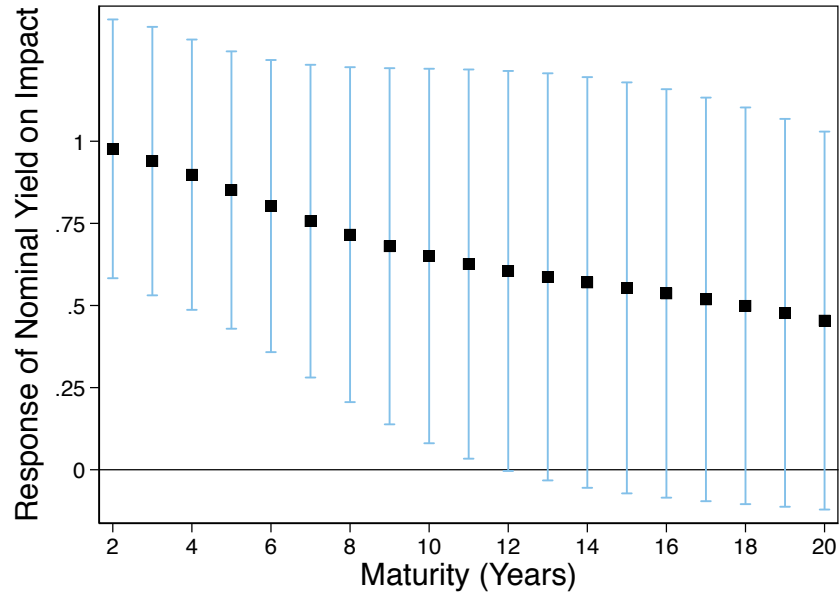
The results are robust to the use of an alternative instrument set. I employ the same instrument set as those discussed in [Gertler and Karadi \(2015\)](#) and [Nakamura and Steinson \(2017\)](#), namely: surprises in the current month's fed funds futures (FF1), in the three month ahead monthly fed funds futures (FF4), and in the six month, nine month and year

ahead futures on three month Eurodollar deposits (ED2, ED3, ED4). Figure (2) plots the estimated results using those alternative instruments: monetary shocks affect interest rate at long horizon.

Figure (3) shows the estimation results for the pre-2008 sample. Figure (4) shows that results are not affected by the use of daily changes in interest rates instead of a two-day window. Figure (5) shows that with or without control variables do not affect the results. This suggests that both the endogenous and exogenous components of monetary surprises affect long-term interest rates, a feature that is consistent with the model presented in this paper. Moreover, monetary shocks affect inflation indexed rates at long horizon, which are proxies for real interest rate (Figure 6).

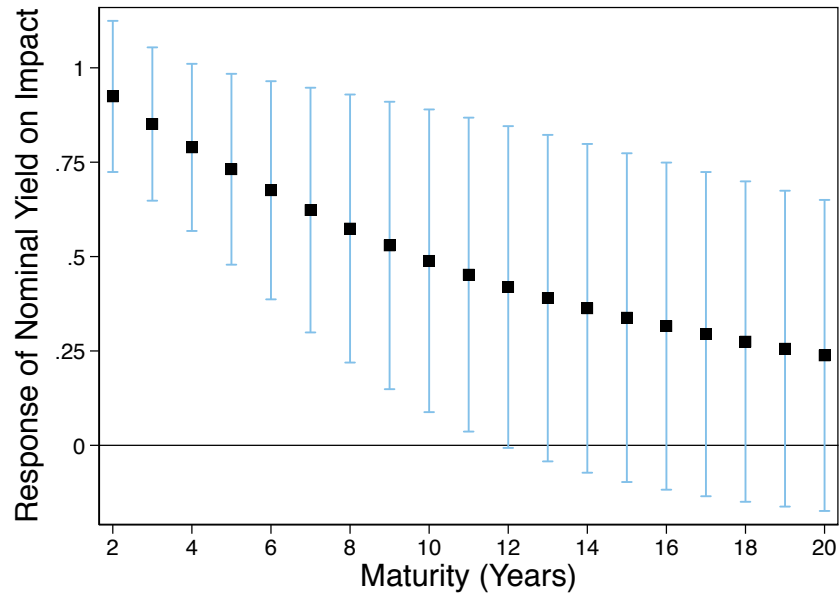
**Monetary Shocks and Forecast Revisions** I conduct a similar set of robustness checks for the puzzles related to monetary shocks and forecast revisions. See Figure (7), Figure (8), and Figure (9) for estimation results. The results are similar to those obtained in the paper.

**Figure 2:** Impact Effect on Yield Robustness Check: G&K Instrument Set



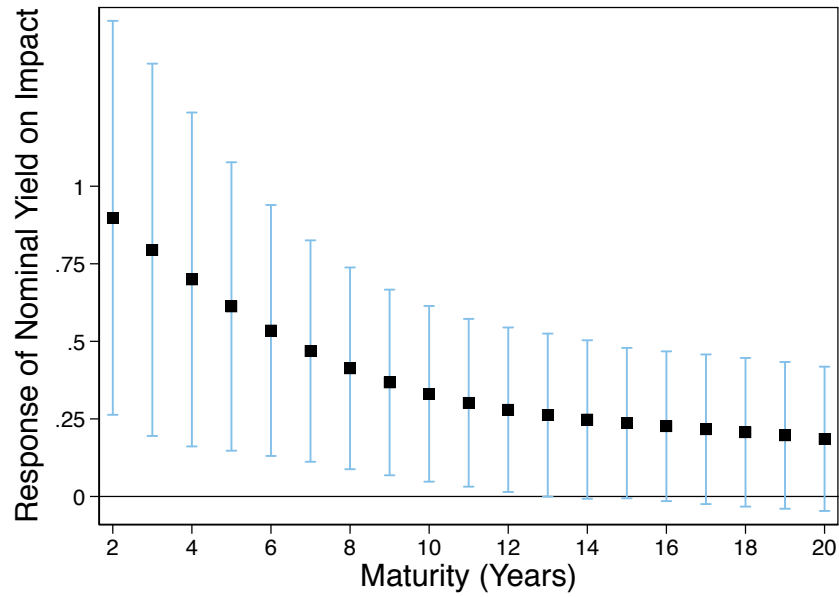
**Notes:** The square dots represent the estimated  $\hat{\beta}^h$  from separated regressions:  $Y_{t+1}^h - Y_{t-1}^h = \alpha^h + \beta^h \Delta MP_t + \gamma^h X_t + \epsilon_{h,t}$  using the same set of HFI monetary surprises discussed in [Gertler and Karadi \(2015\)](#) and [Nakamura and Steinsson \(2017\)](#), namely FF1, FF4, ED2, ED3 and ED4, as instruments for the variable of interest  $\Delta MP_t$ . The first stage F-statistic is equal to 19.6. The blue lines are the 95% confidence interval constructed using standard errors that are robust to serial correlation and heteroskedasticity. Sample: 1990M2 to 2012M12.

**Figure 3: Impact Effect on Yield Robustness Check: Pre-2008 Sample**



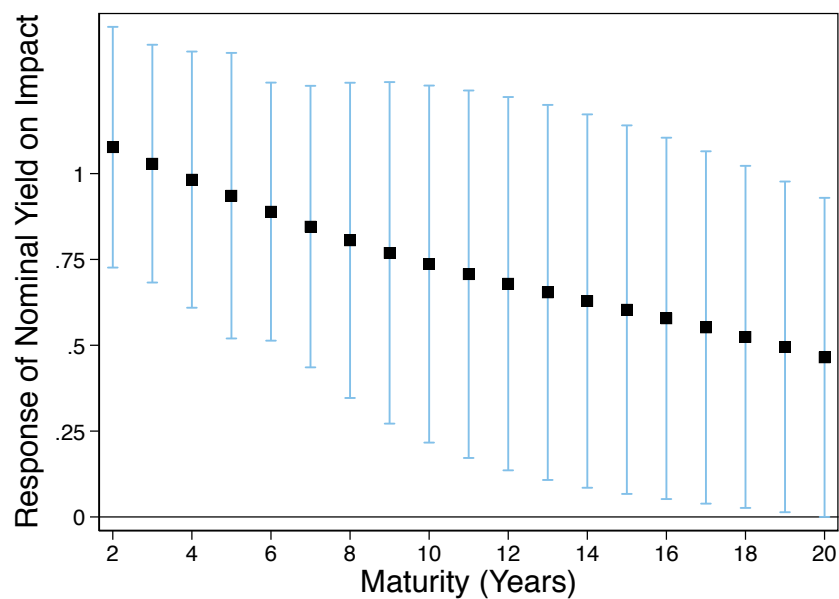
**Notes:** The square dots represent the estimated  $\hat{\beta}^h$  from separated regressions:  $Y_{t+1}^h - Y_{t-1}^h = \alpha^h + \beta^h \Delta MP_t + \gamma^h X_t + \epsilon_{h,t}$  using HFI monetary surprises as instruments for the variable of interest  $\Delta MP_t$ . The first stage F-statistic is equal to 19.6. The blue lines are the 95% confidence interval constructed using standard errors that are robust to serial correlation and heteroskedasticity. Sample: 1990M2 to 2007M12.

**Figure 4:** Impact Effect on Yield Robustness Check: Daily Change



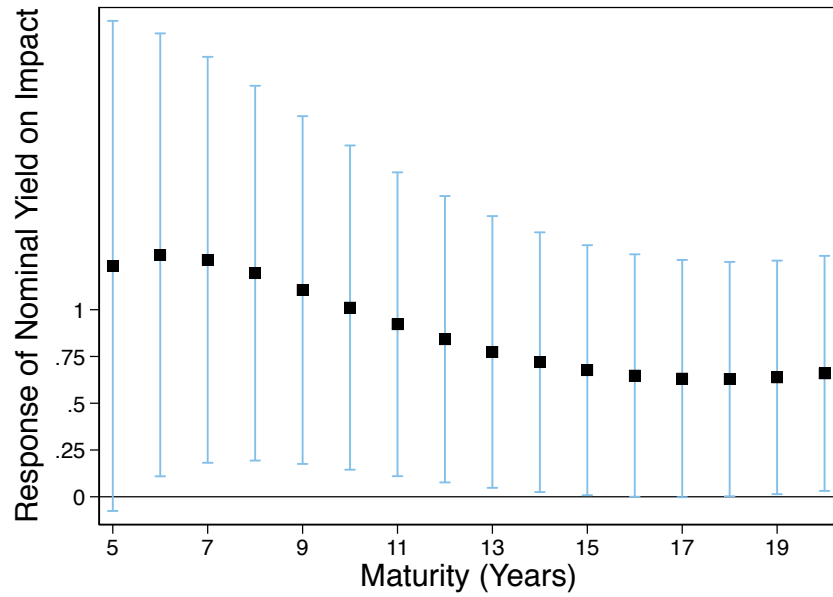
**Note:** The square dots represent the estimated  $\hat{\beta}^h$  from separated regressions:  $Y_t^h - Y_{t-1}^h = \alpha^h + \beta^h \Delta MP_t + \gamma^h X_t + \epsilon_{h,t}$  using HFI monetary surprises as instruments for the variable of interest  $\Delta MP_t$ . The first stage F-statistic is equal to 19.6. The blue lines are the 95% confidence interval constructed using standard errors that are robust to serial correlation and heteroskedasticity. Sample: 1990M2 to 20012M12.

**Figure 5:** Impact Effect on Yield Robustness Check: Excluding Greenbook Forecasts



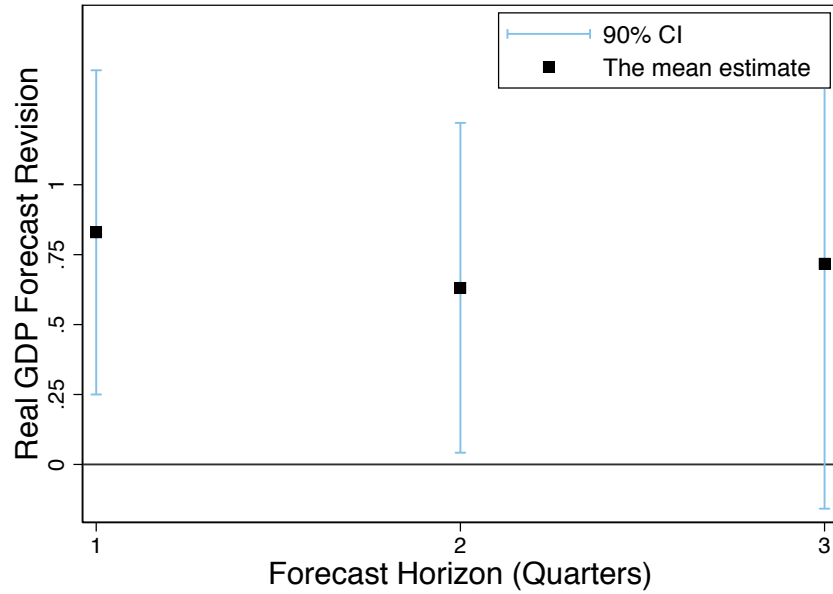
**Note:** The square dots represent the estimated  $\hat{\beta}^h$  from separated regressions:  $Y_{t+1}^h - Y_{t-1}^h = \alpha^h + \beta^h \Delta MP_t + \epsilon_{h,t}$  using HFI monetary surprises as instruments for the variable of interest  $\Delta MP_t$ . The first stage F-statistic is equal to 19.6. The blue lines are the 95% confidence interval constructed using standard errors that are robust to serial correlation and heteroskedasticity. Sample: 1990M2 to 20012M12.

**Figure 6:** Impact Effect on Yield Robustness Check: Inflation Indexed Yields



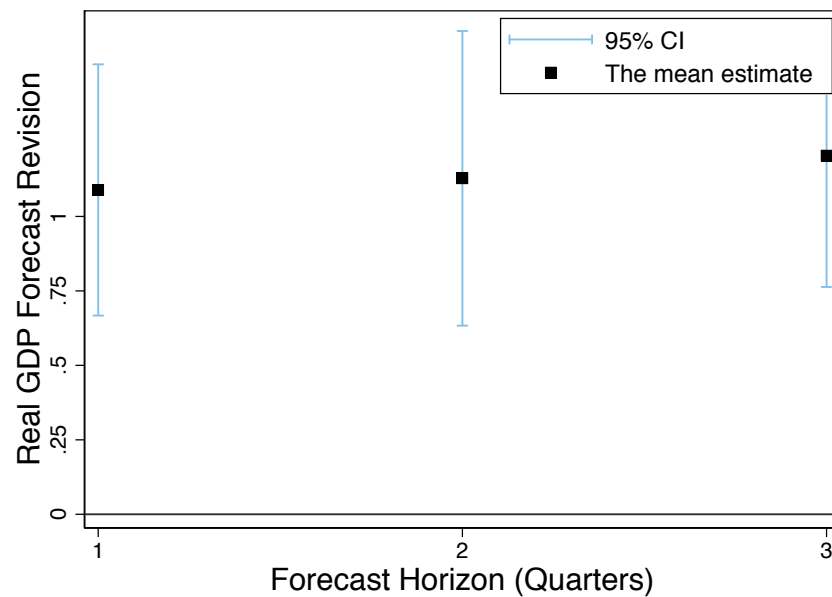
**Note:** The square dots represent the estimated  $\hat{\beta}^h$  from separated regressions:  $Y_{t+1}^h - Y_{t-1}^h = \alpha^h + \beta^h \Delta MP_t + \gamma^h X_t + \epsilon_{h,t}$  using cleaned HFI monetary surprises as instruments for the variable of interest  $\Delta MP_t$ . The first stage F-statistic is equal to 19.6. The blue lines are the 95% confidence interval constructed using standard errors that are robust to serial correlation and heteroskedasticity. Sample: 1990M2 to 20012M12.

**Figure 7:** Impact Effect on Forecast Revision Robustness Check: G&K Instrument Set



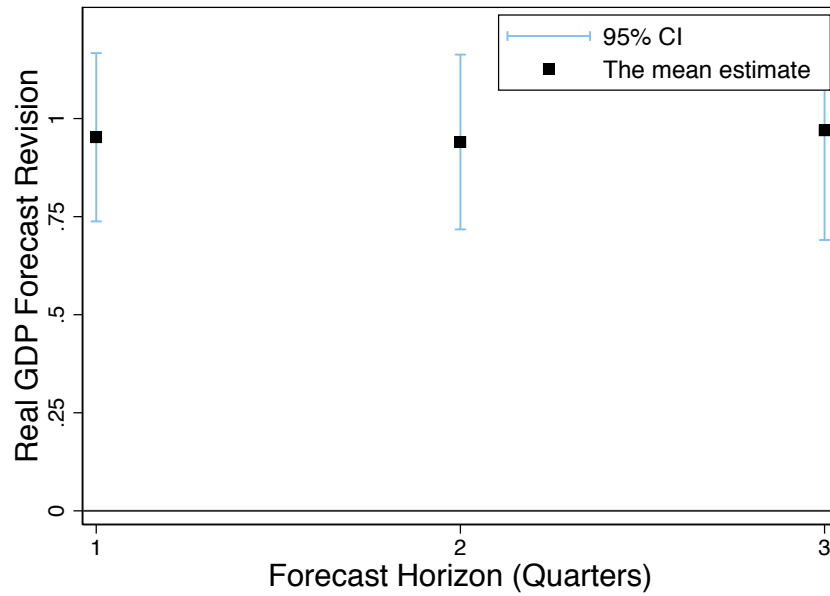
**Note:** The square dots represent the estimated  $\hat{\beta}_j$ s from separated regressions:  $y_{t+j|t} - y_{t+j|t-1} = \alpha^j + \beta^j MP_t + \gamma^j X_t + v_{j,t}$ , using the same set of HFI monetary surprises (cleaned) discussed in [Gertler and Karadi \(2015\)](#) and [Nakamura and Steinsson \(2017\)](#), namely FF1, FF4, ED2, ED3 and ED4, as instruments for the variable of interest  $\Delta MP_t$ . The first stage F-statistic is equal to 8.8. The forecast horizons are denoted in quarters and  $y_{t+j|t}$  denotes the forecast made at time  $t$  about the real GDP at quarter  $t + j$ . The blue lines are the 90% confidence interval constructed using standard errors that are robust to serial correlation and heteroskedasticity. Sample: 1990Q1 to 2012Q4.

**Figure 8:** Impact Effect on Forecast Revision Robustness Check: Pre-2008 Sample



**Note:** The square dots represent the estimated  $\hat{\beta}_j$ s from separated regressions:  $y_{t+j|t} - y_{t+j|t-1} = \alpha^j + \beta^j MP_t + \gamma^j X_t + v_{j,t}$ , using cleaned HFI monetary surprises as instruments for the variable of interest  $\Delta MP_t$ . The first stage F-statistic is equal to 11.7. The forecast horizons are denoted in quarters and  $y_{t+j|t}$  denotes the forecast made at time  $t$  about the real GDP at quarter  $t + j$ . The blue lines are the 95% confidence interval constructed using standard errors that are robust to serial correlation and heteroskedasticity. Sample: 1990Q1 to 2007Q4.

**Figure 9:** Impact Effect on Forecast Revision Robustness Check: Raw HFI Monetary Suprises



**Note:** The square dots represent the estimated  $\hat{\beta}_j$ s from separated regressions:  $y_{t+j|t} - y_{t+j|t-1} = \alpha^j + \beta^j MP_t + \gamma^j X_t + v_{j,t}$ , using raw HFI monetary surprises as instruments for the variable of interest  $\Delta MP_t$ . The first stage F-statistic is equal to 12. The forecast horizons are denoted in quarters and  $y_{t+j|t}$  denotes the forecast made at time  $t$  about the real GDP at quarter  $t + j$ . The blue lines are the 95% confidence interval constructed using standard errors that are robust to serial correlation and heteroskedasticity. Sample: 1990Q1 to 2007Q4.

## C Model Solution

Define:  $X_t^b = (\epsilon_t^m, \hat{g}_t, \delta_t, \epsilon_t^\pi, \hat{i}_{t-1}, \hat{\pi}_{t-1}, u_{t-1}^m)'$  and  $X_t^f = (\hat{y}_t, \hat{\pi}_t)'$ . The model can be summarized as:

$$M^0 \begin{bmatrix} X_{t+1}^b \\ X_{t+1|t}^f \end{bmatrix} = M^1 \begin{bmatrix} X_t^b \\ X_t^f \end{bmatrix} + M^2 \begin{bmatrix} X_{t|t}^b \\ X_t^f \end{bmatrix} + M^3 u_{t+1} \quad (1)$$

or explicitly:

$$M^0 \begin{bmatrix} \epsilon_{t+1}^m \\ \hat{g}_{t+1} \\ \delta_{t+1} \\ \epsilon_{t+1}^\pi \\ \hat{i}_t \\ \hat{\pi}_t \\ u_t^m \\ \hat{y}_{t+1|t} \\ \hat{\pi}_{t+1|t} \end{bmatrix} = M^1 \begin{bmatrix} \epsilon_t^m \\ \hat{g}_t \\ \delta_t \\ \epsilon_t^\pi \\ \hat{i}_{t-1} \\ \hat{\pi}_{t-1} \\ u_{t-1}^m \\ \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} + M^2 \begin{bmatrix} \epsilon_{t|t}^m \\ \hat{g}_{t|t} \\ \delta_t \\ \epsilon_t^\pi \\ \hat{i}_{t-1} \\ \hat{\pi}_{t-1} \\ u_{t-1|t}^m \\ \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} + M^3 \begin{bmatrix} \epsilon_{t+1}^g \\ \epsilon_{t+1}^\delta \\ \epsilon_{t+1}^\pi \\ \epsilon_{t+1}^m \end{bmatrix},$$

where

$$\begin{aligned}
M^0 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \frac{\beta}{1+\omega\beta} \end{bmatrix} & M_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
M^1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_\delta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & (1-\rho_m)\rho_g & (1-\rho_m)\rho_\delta & 0 & \rho_m & 0 & \rho_u & (1-\rho_m)\phi_y & (1-\rho_m)\rho_\pi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & \rho_u & 0 & 0 \\ 0 & 0 & (1-\rho_\delta) & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -\frac{\omega}{1+\omega\beta} & 0 & -\kappa(1+\varphi) & 0 \end{bmatrix} \\
M^2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\rho_g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

Transform (1) into:

$$\begin{bmatrix} X_{t+1}^b \\ X_{t+1|t}^f \end{bmatrix} = A^1 \begin{bmatrix} X_t^b \\ X_t^f \end{bmatrix} + A^2 \begin{bmatrix} X_{t|t}^b \\ X_t^f \end{bmatrix} + A^3 u_{t+1} \quad (2)$$

where  $A^1 \equiv (M^0)^{-1}M^1$ ,  $A^2 \equiv (M^0)^{-1}M^2$  and  $A^3 \equiv (M^0)^{-1}M^3$

Variables that are observable to private agent and relevant for belief updating are summarized in vector  $Z_t$  :

$$Z_t = C \begin{bmatrix} X_t^b \\ X_t^f \end{bmatrix} + v_t \quad (3)$$

or explicitly:

$$\begin{bmatrix} i_t \\ \Delta a_t \\ s_t^p \end{bmatrix} = C \begin{bmatrix} \epsilon_t^m \\ \hat{g}_t \\ \delta_t \\ \epsilon_t^\pi \\ \hat{i}_{t-1} \\ \hat{\pi}_{t-1} \\ u_{t-1}^m \\ \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_t^a \\ \epsilon_t^p \end{bmatrix} \quad (4)$$

where:

$$C = \begin{bmatrix} 1 & (1 - \rho_m)\rho_g & (1 - \rho_m)\rho_\delta & 0 & \rho_m & 0 & \rho_u & (1 - \rho_m)\phi_y & (1 - \rho_m)\rho_\pi \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Take expectation on (2):

$$\begin{bmatrix} X_{t+1|t}^b \\ X_{t+1|t}^f \end{bmatrix} = W \begin{bmatrix} X_{t|t}^b \\ X_{t|t}^f \end{bmatrix} \quad (5)$$

where  $W \equiv A^1 + A^2$ . The Schur decomposition of matrix  $W$  is  $HUH^{-1}$ . Pre-multiply the previous equation by  $H^{-1}$  and define  $Y_t \equiv H^{-1}X_{t|t}$  we get:

$$Y_{t+1} = UY_t \quad (6)$$

Solving the 2nd block of  $Y_{t+1}$  first, where  $|\lambda_i| > 1$ . Eliminating explosive equilibrium

implies:

$$\begin{aligned} Y_{f,t} &= 0 \\ \Rightarrow X_t^f &= GX_{t|t}^b, \end{aligned} \quad (7)$$

where  $G \equiv -(H_{22}^{inv})^{-1}H_{21}^{inv}$ . Note that through out the paper the matrix  $H \equiv \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \equiv \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ , similar notations apply to other matrixes. The 2nd block of (5) together with (7) implies:

$$X_{t+1|t}^b = J^b X_{t|t}^b \quad (8)$$

$$X_{t+1|t}^f = J^f X_{t|t}^b, \quad (9)$$

where  $J^b \equiv (W_{11} + W_{12}G)$ ,  $J^f \equiv G(W_{11} + W_{12}G)$ .

(2) - (5):

$$\begin{bmatrix} X_{t+1}^b - X_{t+1|t}^b \\ 0 \end{bmatrix} = A^1 \begin{bmatrix} X_t^b - X_{t|t}^b \\ 0 \end{bmatrix} + A^3 u_{t+1}$$

The first block implies that:

$$X_{t+1}^b = HX_t^b + JX_{t|t}^b + A_1^3 u_{t+1} \quad (10)$$

where  $J \equiv W_{11} + W_{12}G - A_{11}^1$  and  $H \equiv A_{11}$ .

Notice that in the belief matrix  $X_{t|t}^b$ , only two variables are not perfectly observed by agents, those are  $\epsilon_t^m$ , and  $g_t$ . Denote  $x_t \equiv [\epsilon_t^m, g_t, u_{t-1}^m]'$  and  $x_{t|t} \equiv [\epsilon_{t|t}^m, g_{t|t}, u_{t-1|t}^m]'$ . The remaining variables are perfectly observable, therefore equal to their counter-parts in matrix  $X_t^b$ . Next, we will find law of motion for  $x_{t|t}$ .

Rewrite (3):

$$\begin{aligned} Z_t &= C_1 X_t^b + C_2 X_t^f + v_t \\ &= C_1 X_t^b + C_3 X_{t|t}^b + v_t \\ &= C_b X_t^b + Mx_{t|t} + v_t, \end{aligned}$$

where  $C_3 = C_2 G$ ,  $C_b \equiv C_1 + [0_{3,2} \ C_3(:, 3 : end - 1) \ 0_{3,1}]$ ,  $M \equiv [C_3(:, 1 : 2) \ C_3(:, end)]$ . The above signal equation can be transform to:

$$Z^x = Lx_t + Mx_{t|t} + v_t, \quad (11)$$

where  $Z^x \equiv Z_t - [0_{3,2} \ C_3(:, 3 : end - 1) \ 0_{3,1}]X_t^b$ ,  $L = [C_b(:, 1 : 2) \ C_b(:, end)]$ . Note that:

$$x_t = J^x x_{t-1} + u_t^x, \quad (12)$$

where  $J^x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho_g & 0 \\ 1 & 0 & \rho_u \end{bmatrix}$  and  $u_t^x = \begin{bmatrix} \epsilon_t^m \\ \epsilon_t^g \end{bmatrix}$ . Now, results derived in [Svensson and Woodford \(2003\)](#) applies directly.

$$x_{t|t}^b = x_{t|t-1}^b + K(L(x_t - x_{t|t-1}) + v_t) \quad (13)$$

$$K = PL'(LPL' + \Sigma_{vv})^{-1} \quad (14)$$

$$P = J^x[P - PL'(LPL' + \Sigma_{vv})^{-1}LP]J^{x'} + \Sigma_{ux}^2 \quad (15)$$

**State Space representation**  $X_t \equiv \begin{bmatrix} X_t^b \\ x_{t|t} \end{bmatrix}$  and  $V_t \equiv \begin{bmatrix} \epsilon_t^g \\ \epsilon_t^\delta \\ \epsilon_t^\pi \\ \epsilon_t^m \\ \epsilon_t^a \\ \epsilon_t^p \end{bmatrix}$ . We collect the relevant equations into one state space representation.

$$X_{t+1} = AX_t + BV_{t+1}, \quad (16)$$

$$X_t^f = FX_t \quad (17)$$

$$X_{t+1|t}^f = F_1 X_t \quad (18)$$

with

$$\begin{aligned}
A &= \begin{bmatrix} A_b \\ A_{bb} \end{bmatrix} B = \begin{bmatrix} B_b \\ B_{bb} \end{bmatrix} \\
A_b &= \begin{bmatrix} H & 0_{nb \times 3} \end{bmatrix} + \begin{bmatrix} 0_{nb \times 2} & J(:, 3:end-1) & J(:, 1:2) & J(:, end) \end{bmatrix}, \\
B_b &= A_1^3 L^u, L^u = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \\
A_{bb} &= \begin{bmatrix} 0_{2 \times nb} & J^x - KLJ^x \end{bmatrix} + \begin{bmatrix} KLJ^x(:, 1:2) & 0_{2 \times nb} & KLJ^x(:, end) \end{bmatrix} \\
B_{bb} &= KLL^x + KL^v, L^v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, L^x = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
F &= \begin{bmatrix} 0_{nf \times 2} & G(:, 3:end-1) & 0_{nf \times 1} & G(:, 1:2) & G(:, end) \end{bmatrix} \\
F_1 &= \begin{bmatrix} 0_{nf \times 2} & J^f(:, 3:end-1) & 0_{nf \times 1} & J^f(:, 1:2) & J^f(:, end) \end{bmatrix} \\
F_2 &= \begin{bmatrix} 0_{nf \times 2} & J^{f^2}(:, 3:end-1) & 0_{nf \times 1} & J^{f^2}(:, 1:2) & J^{f^2}(:, end) \end{bmatrix}, J^{f^2} = J^b J^f
\end{aligned}$$

## C.1 Term structure

This section derives term structure implied by the model discussed above. The goal is to write down the term structure in terms of our state vector  $X_t$ .  $i_t$  enters as the fourth element in vector  $X_{t+1}$ , therefore:

$$i_t = M_I X_t, \quad (19)$$

where  $M_I \equiv A(5, :)$ .

The stochastic discount factor, both nominal ( $M_{t+1}^n$ ) and real ( $M_{t+1}^r$ ) can be derived from consumer's first order condition:

$$\begin{aligned}
M_{t+1}^n &= \beta \frac{U_{c,t+1} P_t}{U_{c,t} P_{t+1}} \\
M_{t+1}^r &= \beta \frac{U_{c,t+1}}{U_{c,t}}
\end{aligned}$$

Given the utility function specified in this paper,  $m_{t+1} \equiv \log M_{t+1}$  is defined as:

$$m_{t+1}^n = \log \beta - y_{t+1} + y_t - \pi_{t+1} + (\rho_\delta - 1)\delta_t - g_t \quad (20)$$

$$m_{t+1}^r = \log \beta - y_{t+1} + y_t + (\rho_\delta - 1)\delta_t - g_t \quad (21)$$

Consider the nominal discount factor, rewrite in terms of state variable:

$$m_{t+1}^n = \log \beta + \begin{bmatrix} -1 & -1 \end{bmatrix} X_{t+1}^f + \begin{bmatrix} 1 & 0 \end{bmatrix} X_t^f + \begin{bmatrix} 0 & -1 & \rho_\delta - 1 & 0_{7 \times 1} \end{bmatrix} X_t$$

Recall that  $X_t^f \equiv FX_t$ ,  $X_{t+1}^f = FAX_t + FBV_{t+1}$ , plug them in:

$$m_{t+1}^n = \bar{m} + M_n X_t + M_{vn} V_{t+1}, \quad (22)$$

where  $\bar{m} \equiv \log \beta$ ,  $M_n \equiv \begin{bmatrix} -1 & -1 \end{bmatrix} FA + \begin{bmatrix} 1 & 0 \end{bmatrix} F + \begin{bmatrix} 0 & -1 & \rho_\delta - 1 & 0_{1 \times 5} \end{bmatrix}$  and  $M_{vn} \equiv \begin{bmatrix} -1 & -1 \end{bmatrix} FB$ .

Similarly:

$$m_{t+1}^r = \bar{m} + M_r X_t + M_{vr} V_{t+1} \quad (23)$$

where  $M_r \equiv \begin{bmatrix} -1 & 0 \end{bmatrix} FA + \begin{bmatrix} 1 & 0 \end{bmatrix} F + \begin{bmatrix} 0 & -1 & \rho_\delta - 1 & 0_{5 \times 1} \end{bmatrix}$  and  $M_{vr} \equiv \begin{bmatrix} -1 & 0 \end{bmatrix} FB$ .

The no arbitrage condition:

$$P_t^{n+1} = E_t(M_{t+1} P_{t+1}^n) \quad (24)$$

The relation between yield and price of zero coupon bond:

$$i_t^{(n)} = -n^{-1} \log(P_t^n) \quad (25)$$

With (19) (22) (23) (24) (25) we can derive the yield curve as linear function of state variable  $X_t$ . Let's begin with the nominal yield and the real yield curve is derived similarly. Note that  $i_t^{(1)} \equiv i_t$ . From (19) (25) and let  $n = 1$  we get:

$$\begin{aligned} P_{t+1}^1 &= \exp(-i_{t+1}) \\ &= \exp(-M_I X_{t+1}) \\ &= \exp(-M_I A X_t - M_I B V_{t+1}) \end{aligned} \quad (26)$$

Plug (23) and (26) into (25):

$$\begin{aligned}
P_t^2 &= v_{h,t}(M_{t+1}P_{t+1}^1) \\
&= v_{h,t}[\exp(\bar{m} + M_n X_t + M_{vn} U_{t+1} - M_I A X_t - M_I B V_{t+1})] \\
&= v_{h,t}[\exp(\bar{m} + (M_n - M_I A) X_t + (M_{vn} - M_I B) V_{t+1})]
\end{aligned}$$

Under the assumption that shocks are normally distributed, the previous equation follows a log-normal distribution<sup>1</sup>. Therefore:

$$P_t^2 = \exp(\bar{m} + (M_n - M_I A)DX_t + 0.5Var_2), \quad (27)$$

we have used the fact that  $X_{t|t} = DX_t$  with  $D \equiv$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \cdot Var_2$$

depends on parameters and are remained to be determined. By (25)

$$\begin{aligned}
i_t^{(2)} &= -0.5 \log(P_t^2) \\
&= -0.5\bar{m} - 0.25Var_2 - 0.5(M_n - M_I A)DX_t
\end{aligned} \quad (28)$$

We have derived the expression for yield of a zero-coupon bond with maturity of 2. We will show in general for  $n \geq 2$ :

$$\log P_t^n = A_n + B_n X_t \quad (29)$$

---

<sup>1</sup>If  $x$  follows a log-normal distribution,  $E(e^x) = e^{E(x) + 0.5\sigma_x^2}$

For  $n = 1$ ,  $\log P_t^1 = -M_I X_t$ . And we have shown for  $n = 2$ :

$$\begin{aligned} A_2 &= \bar{m} + 0.5Var_2 \\ B_2 &= (M_n - M_I A)D \end{aligned}$$

In order to find the general rule, let's take one more step and set  $n = 3$ :

$$P_t^3 = v_{h,t}(M_{t+1}P_{t+1}^2)$$

(29) implies that  $P_{t+1}^2 = \exp(A_2 + B_2 A X_t + B_2 B V_{t+1})$ . Therefore:

$$\begin{aligned} P_t^3 &= v_{h,t}[\exp(\bar{m} + M_n X_t + M_{vn} V_{t+1} + A_2 + B_2 A X_t + B_2 B V_{t+1})] \\ &= v_{h,t}[\exp(\bar{m} + A_2 + (M_n + B_2 A)X_t + (M_{vn} + B_2 B)V_{t+1})] \end{aligned}$$

Hence:

$$\log P_t^3 = A_3 + B_3 X_t,$$

with

$$\begin{aligned} A_3 &= \bar{m} + A_2 + 0.5Var_3 \\ B_3 &= (M_n + B_2 A)D \end{aligned}$$

The recursive relations for  $A_n$  and  $B_n$  are:

$$A_n = A_{n-1} + \bar{m} + 0.5Var_n \quad (30)$$

$$B_n = (M_n + B_{n-1}A)D \quad (31)$$

The expressions for  $Vars$  are:

$$Var_2 = [B_2 D^{-1}]^b P[B_2 D^{-1}]^{b'} + (M_{vn} + M_I B) \Sigma_{VV} (M_{vn} + M_I B)' + \sigma^2 \Psi^2(\sigma_g^2 + \sigma_a^2 + \sigma_m^2)$$

$$Var_3 = [B_3 D^{-1}]^b P[B_3 D^{-1}]^{b'} + (M_{vn} + B_2 B) \Sigma_{VV} (M_{vn} + B_2 B)' + \sigma^2 \Psi^2(\sigma_g^2 + \sigma_a^2 + \sigma_m^2)$$

$$Var_n = [B_n D^{-1}]^b P[B_n D^{-1}]^{b'} + (M_{vn} + B_{n-1} B) \Sigma_{VV} (M_{vn} + B_{n-1} B)' + \sigma^2 \Psi^2(\sigma_g^2 + \sigma_a^2 + \sigma_m^2)$$

Thus the yield curve with different maturities can be collected in:

$$\begin{bmatrix} i_t \\ i_t^2 \\ \cdot \\ \cdot \\ i_t^n \end{bmatrix} = \begin{bmatrix} -A_1 \\ -\frac{1}{2}A_2 \\ \cdot \\ \cdot \\ -\frac{1}{n}A_n \end{bmatrix} + \begin{bmatrix} -B_1 \\ -\frac{1}{2}B_2 \\ \cdot \\ \cdot \\ -\frac{1}{n}B_n \end{bmatrix} X_t + e_t^i \quad (32)$$

## D Bayesian Estimation

**Measurement equations** The measurement equations are:

$$\log(RGDP_t/POP_t) - \log(RGDP_{t-1}/POP_{t-1}) = g^* + \hat{y}_t - \hat{y}_{t-1} + \hat{g}_t + \epsilon_t^a$$

$$Inflation_t = \hat{\pi}_t + \pi^*$$

$$Y2Y_t/4 = -\frac{1}{8}B_8X_t + e_t^{y2}$$

$$Y5Y_t/4 = -\frac{1}{20}B_{20}X_t + e_t^{y5}$$

$$Y10Y_t/4 = -\frac{1}{40}B_{40}X_t + e_t^{y10},$$

where  $RGDP_t/POP_t$  denotes the per capita real GDP, Inflation is the GDP price deflator,  $Y2Y_t$ ,  $Y5Y_t$  and  $Y10Y_t$  are the demeaned nominal yields with maturity of two, five and ten years respectively. By demeaning the yield I forgo testing the model's ability to match the average slope of yield curve. Matching the empirical slope of the yield curve would require additional features such as liquidity premium, which is beyond the scope of this paper. Note that I have add measurement errors to the yields to allow for potential time varying risk premium that are not captured in the model.

**Posterior Convergence Check** Figure (10) provides a visual representation of posterior distributions. Both the trace plots (see Figure (11)) and the recursive average of parameters (see Figure (12)) suggest the convergence of posterior distributions.

Figure 10: Posterior Distributions

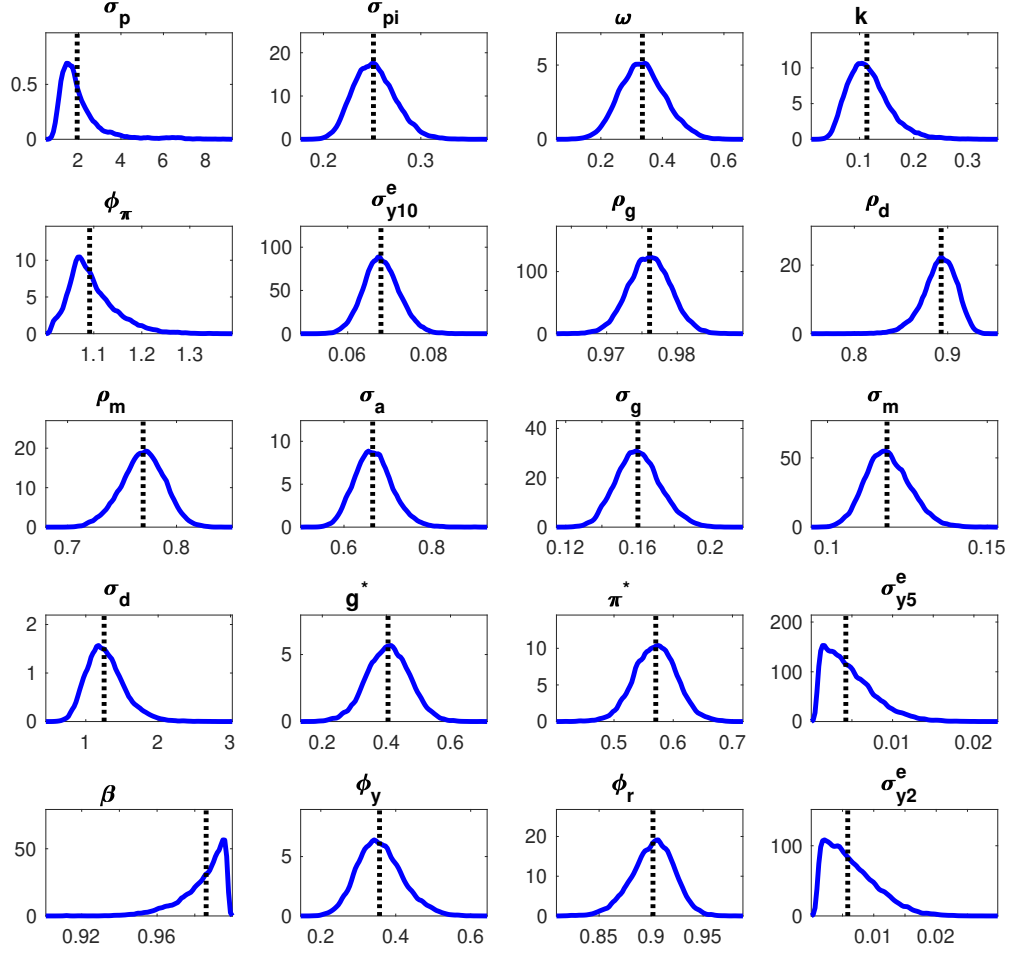
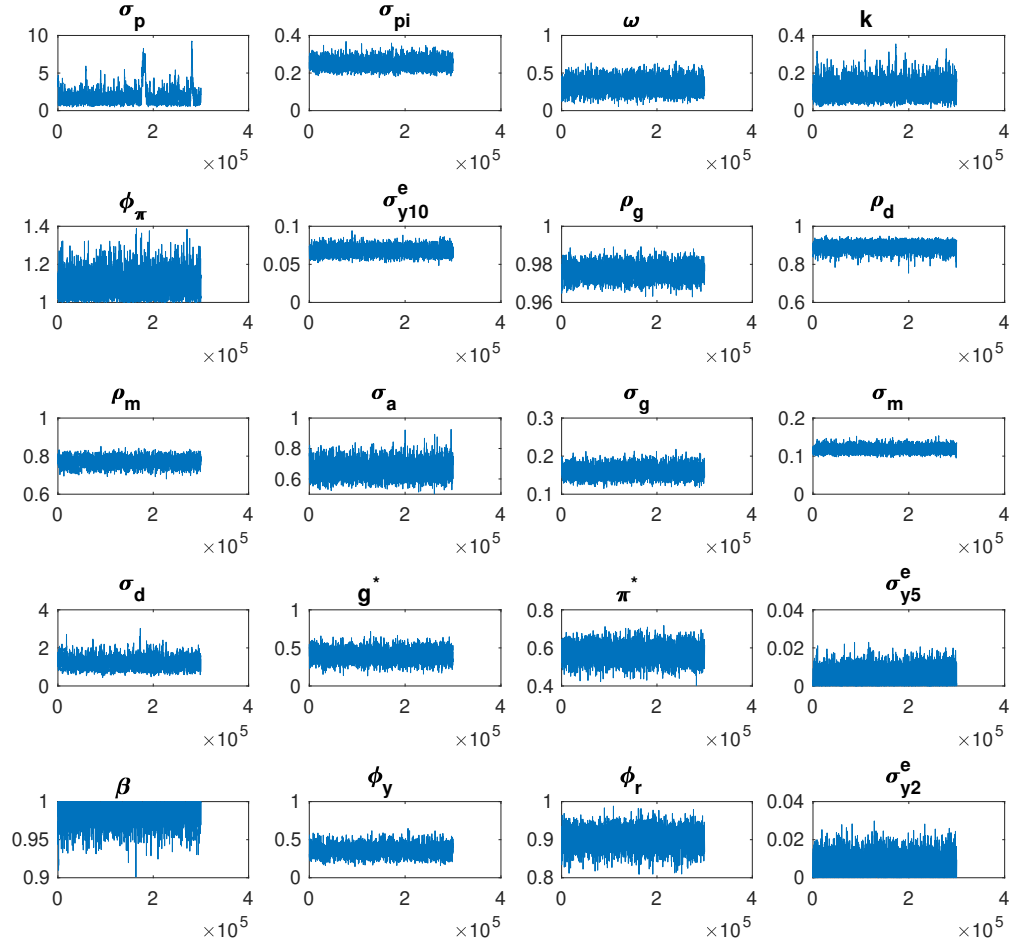
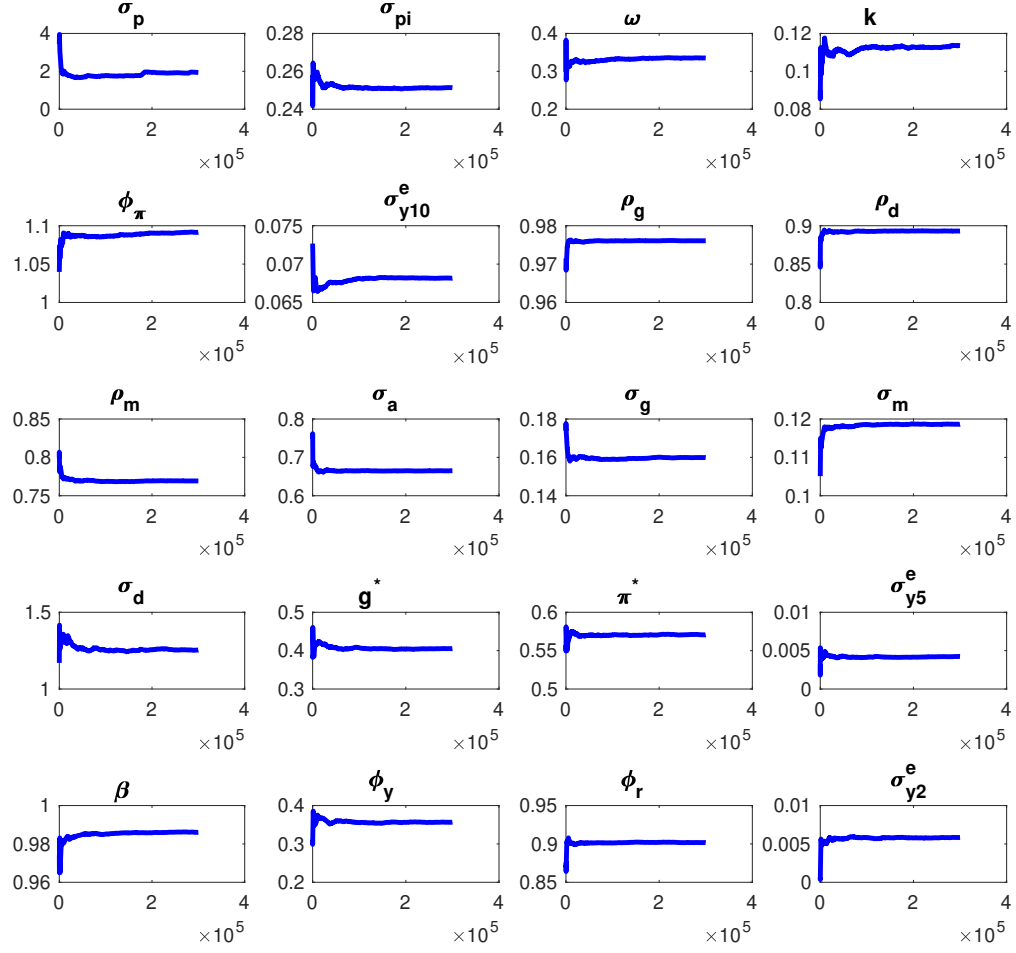


**Figure 11: Convergence Check I: Traceplots**



**Note:** A traceplot is a plot of the value of the draw of the parameter at each iteration against the iteration number

**Figure 12: Convergence Check II: Recursive Average**



## References

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