

Bubbly Credit Cycles^{*}

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Abstract

Canonical macroeconomic and financial models require credit to be equal to its fundamental component, i.e., the net present value of the net flows to creditors. Per this conventional view, credit booms are expected to precede increased flows to creditors. However, data suggests otherwise. To rationalize the novel empirical findings, we develop a model with financial frictions and heterogeneous firms, allowing for the existence of credit bubbles. We show that a positive credit bubble shock raises the aggregate credit and output while depressing the credit's fundamental component through firms' precautionary savings.

Keywords: Credit Bubbles, Financial Frictions, Heterogeneous Firms

JEL Classification: D92, E32, E44

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1 Introduction

Credit booms and busts accompanied by business cycles are reoccurring global events. These events motivate expanding theoretical literature to uncover the mechanism for credit cycles. While previous studies differ in many dimensions, they typically share the following consensus view: credit is a promise of *future* net outflows to creditors in exchange for *current* net inflows. Built on this paradigm, canonical models feature obtained credit equal to its fundamental component—the net present value (NPV) of future net flows to creditors. It follows that credit booms are expected to be explained by increases in the fundamental component and precede rises in the net flows to creditors.

This paper documents novel evidence that a credit boom is *not* followed by increases in future net flows to creditors. More specifically, the net flows to creditors drop in the short run before converging back to their initial level. Moreover, interest rates barely change. We develop a quantitative model with financial frictions and heterogeneous firms to interpret the empirical findings. Importantly, the model features credit bubbles, i.e., firms can obtain some credits *without* exchanging future net outflows to lenders.¹ Instead, future repayments to creditors are financed through future credits. These credits are referred to as credit bubbles or indefinite rolled-over credits interchangeably. We show that financial frictions, firm heterogeneity, and the newly introduced credit bubbles can reconcile the empirical findings.

The empirical findings are constructed on the basis of a vector autoregressive (VAR) model using aggregate data from the Financial Accounts of the United States for all corporate and non-corporate non-financial firms. The estimated VAR provides estimates for the effects of an *innovation* on the total liabilities on real GDP and, importantly, on *future* net flows to creditors.² In the baseline specification, we rely on credit cycles generated by max-share innovations (see, e.g., Uhlig 2003 and Barsky and Sims 2011) to the total liabilities. Specifically, we compute *the innovation* as the linear combination of reduced-form residuals that maximizes the expected fluctuations in the total liabilities at the horizons corresponding to the business cycle frequency. A positive innovation that increases the total liabilities *today* is accompanied by an expansion in aggregate production. The net flow to creditors *drops* in the subsequent periods before converging to its steady-state value—

¹We model credit bubbles in the spirit of Tirole (1985). See also Domeij and Ellingsen (2018), Farhi and Tirole (2012), Ikeda and Phan (2019), and Martin and Ventura (2012) for some recent examples of rational bubbles along this line in the literature.

²We consider the total liabilities in the data. The total liabilities are equal to the total debt in the model. We use total liabilities and total debt interchangeably whenever the distinction between the two is clear.

the **Fact 1**. The observation that the credit boom is *not* associated with future increases in net flows to creditors needs to be emphasized. Real interest rates are not affected. In addition, the paper presents evidence that the puzzling baseline empirical findings are unlikely to be driven by statistical issues.

A linearized aggregate debt decomposition equation, together with the impulse response functions estimated using the VAR, enable the computation of the responses of the approximated fundamental component of the total liabilities in the spirit of [Galí and Gambetti \(2015\)](#). Following a credit boom, the constructed fundamental component *decreases* significantly both economically and statistically—the **Fact 2**.

The baseline empirical findings are constructed based on reduced-form innovations. The evidence suggests the presence of shocks that generate the puzzling relationship between current total liabilities and future flows to creditors. Documenting the empirical evidence we emphasize does not require that structural shocks be identified. Nevertheless, we present two structural shocks that generate empirical findings similar to those generated at the baseline. The first structural shock is the excess bond premium (EBP) shock constructed by [Gilchrist and Zakrajšek \(2012\)](#), which measures credit supply changes orthogonal to firms’ fundamentals. The second is labeled non-fundamental shocks in the credit market. We identify non-fundamental shocks in the credit market as the shocks that, once shocks to the interest rate are controlled for, maximize the forecast error variance decomposition of the liability-fundamental differential in the subsequent periods. These two structural shocks, independently, generate credit booms not followed by future increases in net flows to creditors. Our paper provides an alternative structural interpretation of [Gilchrist and Zakrajšek \(2012\)](#) EBP shocks. In the following, we argue that these two structural shocks may capture sentiment shocks—shocks to the credit bubble—that could potentially explain the puzzling credit cycle dynamics.

We then build a model to provide a structural interpretation of the newly uncovered facts. The model features standard elements of a workhorse model of firm heterogeneity and financial frictions. Our key innovation is relaxing the conventionally imposed no-Ponzi-game condition, i.e., credit bubbles exist. Firms cannot take an infinite amount of credit bubbles. Instead, the maximum amounts of the credit bubbles are exogenously determined by market sentiment. Furthermore, credit bubbles are consistent with rational expectations. The remaining fraction of debts—the traditional debts defined in the literature—can be rolled over but not indefinitely.

We propose a novel financial shock called a credit bubble shock (or a rollover shock),

defined as an unexpected change (driven by market sentiment) in the size of credit bubbles. Suppose the current market sentiment permits a firm to receive x from financial intermediaries to roll over the bubble component of its debt. Following a rollover shock of the size ϵ , the firm receives $x + \epsilon$ from the financial intermediaries. A positive (negative) ϵ implies an increase (a loss) in the debts to be indefinitely rolled over.

Credit bubble shocks affect the fundamental component of total debt through two channels. On the one hand, a positive rollover shock is equivalent to an increase in firms' net worth, which reduces firms' default probability and, therefore, the borrowing cost. This channel induces firms to increase the fundamental component of credit.

On the other hand, a positive bubble shock can induce firms to reduce the fundamental component due to the precautionary motive. More specifically, credit bubbles are fully backed by the expectation of future credit bubbles rather than future net flows to creditors. Unlike the credit fundamental component, increasing credit bubbles does not entail future net flows to creditors and does not raise default risk. Following a positive credit bubble shock, firms can protect themselves against non-insurable idiosyncratic shocks by *substituting* the credit fundamental component with credit bubbles — the *precautionary saving* channel. In contrast to a standard model of precautionary saving, this model features firms self-insuring by borrowing more (through credit bubbles) rather than saving more.

For a firm with moderate capital stock and net worth, the investment expansion motive is overwhelming, and therefore, a positive rollover shock increases the fundamental component. In contrast, for a firm with relatively abundant capital and net worth, the precautionary motive of self-insurance overrides the motive to expand investment. Thus, a positive rollover shock reduces the fundamental component for those firms. The overall effect of the proposed financial shock on the aggregate amount of fundamental components depends on the *distribution* of firms.

We calibrate the model to match firms' life cycles and financial characteristics in the United States. The model predicts that a positive rollover shock stimulates aggregate output and TFP. More importantly, unlike other fundamental shocks, a bubble shock in our model generates a credit boom followed by *declining* net flows to creditors and fundamental credit consistent with Facts 1 and 2.³

³The other fundamental shocks considered in the paper includes shocks to the collateral values, productivity, and news shocks (an anticipated increase in productivity).

Literature Review The paper contributes to a growing literature on rational bubbles. Like our study, Caballero and Krishnamurthy (2006), Farhi and Tirole (2012), Martin and Ventura (2012), Ikeda and Phan (2019), and Asriyan et al. (2021) model bubbles as intrinsically useless assets (pyramid schemes) and argue that bubbles can potentially relax financial constraints and stimulate investment and output. A different yet closely related literature models asset bubbles as liquidity premium or safety premium, see, e.g., Kocherlakota (2009), Hirano and Yanagawa (2017), Miao and Wang (2018), Biswas et al. (2020), and Dong and Xu (2022). This article is related to works on *government debt bubbles*, e.g., Hellwig and Lorenzoni (2009), Blanchard (2019), Domeij and Ellingsen (2018, 2021), and Kocherlakota (2023), who model public debt bubbles as rollover of debts.⁴ Martin and Ventura (2016) study credit bubbles in a framework with a representative firm. The current paper is the first to present evidence on the co-movement between aggregate credit and the fundamental component. Additionally, it is the first to develop a model with firm dynamics, financial frictions, and credit bubbles. The model introduces a hitherto unexplored mechanism that arises from firm *heterogeneity* and *precautionary saving*. Due to these features, positive shocks in the non-fundamental credit crowd out fundamental credit while leading to a prolonged increase in output.⁵

Our paper contributes to the extensive literature on credit cycles. The extant empirical works document booms and busts in credit markets: see, for example, Claessens et al. (2012), Covas and Haan (2011), Schularick and Taylor (2012), and Müller and Verner (2021). The theoretical line of research seeks to account for the procyclicality of credit and the magnitude of credit fluctuations by investigating the salience of financial frictions and financial shocks, examples of which include Jermann and Quadrini (2012), Gorton and Ordonez (2014), Azariadis et al. (2016), Guerrieri and Lorenzoni (2017), Perri and Quadrini (2018), Kiyotaki and Moore (2019), Gertler et al. (2020), Cui and Kaas (2021), and Asriyan et al. (2022). Those previous theoretical works exclude credit bubbles and feature credit fundamental components as the only component of credits. Hence, these models imply that credit fluctuations are completely aligned with the fundamental component. Our model provides a new interpretation of financial shocks highlighted by, e.g., Jermann and Quadrini (2012), and more importantly, it reconciles the novel findings about

⁴See also related works by Reis (2021), Brunnermeier et al. (2022), and Miao and Su (2023), who model public debt overvaluation as an outcome of safety premium or liquidity premium.

⁵These results are not present in a representative firm framework, as seen in, for example, Martin and Ventura (2016). It can be shown that in their model, the fundamental credit component is always procyclical and moves alongside credit bubbles when the economy is financially constrained.

the co-movement between aggregate credit and the fundamental component.

Two closely related works are [Azariadis et al. \(2016\)](#) and [Tang and Zhang \(2022\)](#). In [Azariadis et al. \(2016\)](#), the authors argue that the procyclicality of credit is primarily driven by self-fulfilling fluctuations in unsecured credit. They note that unsecured credit is *similar* to credit bubbles, as the size of unsecured credit depends on market sentiment. However, unlike the credit bubbles in our paper, the unsecured credit in [Azariadis et al. \(2016\)](#) is part of the fundamental credit because it is backed by expected future net flows to creditors. As a result, the empirical evidence emphasized in our paper remains puzzling within [Azariadis et al. \(2016\)](#)'s framework.

[Tang and Zhang \(2022\)](#) examine equity price bubbles within a framework of competitive markets and heterogeneous firms. They focus on bubbles occurring in a frictionless equity market, rather than a frictional credit market as in our paper. Consequently, [Tang and Zhang \(2022\)](#) does not consider the financial characteristics of firms' debts, such as leverage ratio and spreads, nor do they feature the precautionary motive discussed in our paper. Furthermore, the empirical analysis conducted by [Tang and Zhang \(2022\)](#) suggests that dividends, representing net outflows to shareholders, increase following an upswing in the non-fundamental component of equity value.⁶ In contrast, our paper documents the opposite pattern for firms' debts.

Last but not least, our paper contributes to a vast body of research on heterogeneous firms in macroeconomics and finance literature. Recent works focus on the relationship between firm financing behavior and the transmission of business cycles. Some examples of those works are [Gilchrist et al. \(2014\)](#), [Zetlin-Jones and Shourideh \(2017\)](#), [Crouzet \(2018\)](#), [Begenau and Salomao \(2019\)](#), [Jeenas \(2019\)](#), [Albert and Caggese \(2021\)](#) and [Alfaro et al. \(2021\)](#). Furthermore, works by, for example, [Baker et al. \(2003\)](#), [Eisfeldt and Muir \(2016\)](#), and [Armenter and Hnatkovska \(2017\)](#), document the precautionary saving of firms. More specifically, our paper builds upon the theoretical models developed by [Khan and Thomas \(2013\)](#), [Arellano et al. \(2019\)](#), and [Ottonello and Winberry \(2020\)](#). The current paper contributes to this line of investigation by incorporating credit bubbles and illustrating how firm heterogeneity affects the propagation of rollover shocks that ultimately generate the puzzling reduced-form empirical evidence.

The remainder of the paper is organized as follows. Section 2 presents our empirical findings. Section 3 develops the full model. Section 4 provides the predictions of the model. Section 5 concludes.

⁶See the online appendix for [Tang and Zhang \(2022\)](#).

2 Empirical Analysis

Section 2.1 develops a simple framework that guides the empirical analysis. Section 2.2 documents the data used in the paper. Section 2.3 discusses the empirical strategy. Section 2.4 presents the baseline empirical findings. Section 2.5 addresses potential statistical issues. Section 2.6 provides additional evidence relying on structural shocks.

2.1 A Simple Framework

Our starting point is considering the cash flow (budget constraint) equation of the aggregate non-financial firms at t :

$$R_t D_{t-1} = \underbrace{Gross\ Saving_t + \Delta Equity_t - Invest_t}_{S_t} + D_t, \quad (1)$$

where, D_t is the real market value of total non-financial firms' debt held by the market at the end of period t , R_t denotes the average interest rate associated with D_{t-1} , and S_t denotes the surplus of non-financial firms: gross saving ($Gross\ Saving_t$) plus changes in the market value of total equity ($\Delta Equity_t$) minus total investment ($Invest_t$). Note that the total investment includes both capital and financial investments. Appendix 2.2 derives the Equation (1), allowing for default, and shows that it holds in a market with bonds of different maturities. Empirical measures of these variables are introduced in Section 2.2

Equation (1) can be interpreted as the following. Consider D_t as one-period debts. Firms can repay their old debts and the associated interest payments ($R_t D_{t-1}$) either by creating their surpluses ($S_t > 0$) and/or by taking a new debt ($D_t > 0$). The latter is called *debt rollover*.

By moving D_t to the left hand side of the Equation (1), it is straightforward to see that surpluses are equivalent to net flows to creditors ($R_t D_{t-1} - D_t$):

$$\underbrace{R_t D_{t-1} - D_t}_{\text{Net outflows to creditors}} = \underbrace{S_t}_{\text{Surpluses}} \quad (2)$$

Henceforth, we use the terms surpluses and flows to creditors interchangeably. Equation (2) presents another way of understanding the cash flow equation. Increased debts ($D_t > R_t D_{t-1}$) necessarily means firms are running deficits ($S_t < 0$). On the contrary, by running surpluses at time t , firms repay old debts using their profits. Firms can also repay

old debts by running zero surpluses, provided that they borrow new debts to repay the old ones ($D_t = R_t D_{t-1}$).

Substituting Equation (1) forward obtains:

$$D_t = E_t \sum_{h=1}^{\infty} \left(\prod_{j=1}^h R_{t+j}^{-1} \right) S_{t+j} + \lim_{h \rightarrow \infty} E_t \left(\prod_{j=1}^h R_{t+j}^{-1} \right) D_{t+h}. \quad (3)$$

The above equation can be re-written as:

$$D_t = F_t + D_t^{\infty}, \quad (4)$$

where

$$F_t \equiv E_t \sum_{h=1}^{\infty} \left(\prod_{j=1}^h R_{t+j}^{-1} \right) S_{t+j} \quad (5)$$

$$D_t^{\infty} \equiv \lim_{h \rightarrow \infty} E_t \left(\prod_{j=1}^h R_{t+j}^{-1} \right) D_{t+h}. \quad (6)$$

F_t denotes the fundamental component of the debt backed by future surpluses. The literature typically assumes $D_t^{\infty} = \lim_{h \rightarrow \infty} E_t \left(\prod_{j=1}^h R_{t+j}^{-1} \right) D_{t+h} = 0$, known as the no-Ponzi-game condition. Without imposing this assumption, credit bubbles arise. We use the terms ‘credit bubble’ and ‘indefinite rollover’ interchangeably throughout the paper to describe D_t^{∞} . As its equation indicates, credit bubbles represent debts that can be repaid using future debts indefinitely.

We now re-state the objective of the empirical part of the paper. The existing literature (see e.g., [Jermann and Quadrini 2012](#), and [Azariadis et al. 2016](#)) typically assumes that $D_t = F_t$. Per this view, the acquisition of new credit today should be accompanied by increasing surpluses or declining interest rates in the future. We assess this assumption in the data.

2.2 Data

We use data from the Financial Accounts of the United States, previously known as the Flow of Funds Accounts. The Financial Accounts are parts of the US system of national economic accounts that record the acquisition of assets throughout the US economy, document the sources of the funds used to acquire those assets, and measure the value of

total assets and liabilities. We combine Flows of Funds data for non-financial corporate businesses and non-financial non-corporate businesses.

The total liabilities are the sum of the total liabilities for non-financial corporate and non-financial non-corporate firms. The construction of the surplus (S_t) is straightforward, given the data on gross saving, total investments, and changes in equities. The total investment ($invest_t$) includes total capital expenditures and the net acquisition of financial assets. The gross saving includes profit and foreign earnings retained abroad less net capital transfers paid. The constructed S_t measures net flows to creditors holding the total liabilities of non-financial corporate and non-corporate firms in the U.S. Appendix 2.1 presents detailed explanations about the data used in this paper and their sources. Figure A.18 in the appendix plots the data we use from the Financial Accounts of the United States.

As mentioned earlier, an alternative measure of net flow to creditors is $R_{t+1}D_t - D_{t+1}$. This measure requires information on interest payments only available at an annual frequency. Nevertheless, our findings are robust to the use of this alternative measure. The detailed discussion is postponed to Section 2.5.

2.3 Empirical Strategy

This paper relies on the log-linearized version of Equation (1) to guide the empirical studies:

$$\hat{d}_t = (1 - \rho)\hat{s}_{t+1} + \rho\hat{d}_{t+1} - \hat{r}_{t+1}, \quad (7)$$

where $\hat{x}$ denotes the log deviation of the variable X from its balanced growth path, $\rho \equiv \frac{1}{sd+1}$, and sd denotes the surplus-to-debt ratio on the balanced growth path (BGP).

We consider a VAR model consisting of the following variables: real GDP (y_t), total liabilities (d_t), surplus ($\tilde{s}_t$), and real interest rate (r_t) constructed using Moody's Baa corporate bond yield. All nominal variables are deflated by the GDP deflator. To interpret the results in percentage terms, real GDP and the total liabilities enter the VAR in logarithm. The surplus is transformed into a percentage deviation from its sample mean as a fraction of its mean in absolute value because it contains negative values. That is, $\tilde{s}_t \equiv \frac{S_t - S}{|S|}$, where S represents the sample mean of S_t .

Let $Y_t \equiv [d_t, \tilde{s}_t, y_t, r_t]'$, the reduced-form VAR model is:

$$Y_t = B(L)Y_t + U_t, \quad (8)$$

where L is the lag operator, $B(L)$ is the matrix of lag order polynomials, and $U_t \equiv [u_t^d, u_t^s, u_t^y, u_t^r]$ contains the reduced-form residuals.

The VAR model can be rewritten in the companion form:

$$\mathbb{Y}_t = \mathbb{A}\mathbb{Y}_{t-1} + \mathbb{U}_t,$$

where

$$\mathbb{Y}_t \equiv \begin{bmatrix} Y_t \\ \vdots \\ Y_{t-p+1} \end{bmatrix}, \mathbb{A} \equiv \begin{bmatrix} B_1 & B_2 & \dots & B_{p-1} & B_p \\ I_K & 0 & & 0 & 0 \\ 0 & I_K & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_K & 0 \end{bmatrix}, \mathbb{U}_t \equiv \begin{bmatrix} U_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

K and p denote the number of variables and the number of lags, respectively.

The VAR model can inform us, on average, what will happen in the future, given the current economic state. Formally, given the parameter estimates ($\hat{B}$ s) and the data available at time t (Y_t), the $E_t Y_{t+h}$ can be computed as:

$$E_t(Y_{t+h}|Y_t) = \mathbb{A}^h Y_t.$$

For a given choice of reduced-form residuals, the average changes in Y at h periods ahead ($E_t Y_{t+h}$)—impulse response functions (IRFs)—can be easily computed.

The central objective of the empirical part is to document what happens to future surpluses ($E_t s_{t+h}$) when the current level of debt (d_t) increases. Under rational expectations, the impulse response of s_{t+h} to a shock at time t captures $E_t[s_{t+h}]$ conditional on the shock.

In the baseline, we consider the max-share innovations, which are constructed as the linear combination of reduced-form residuals that maximizes the fluctuations in d_t at the business cycle frequency.

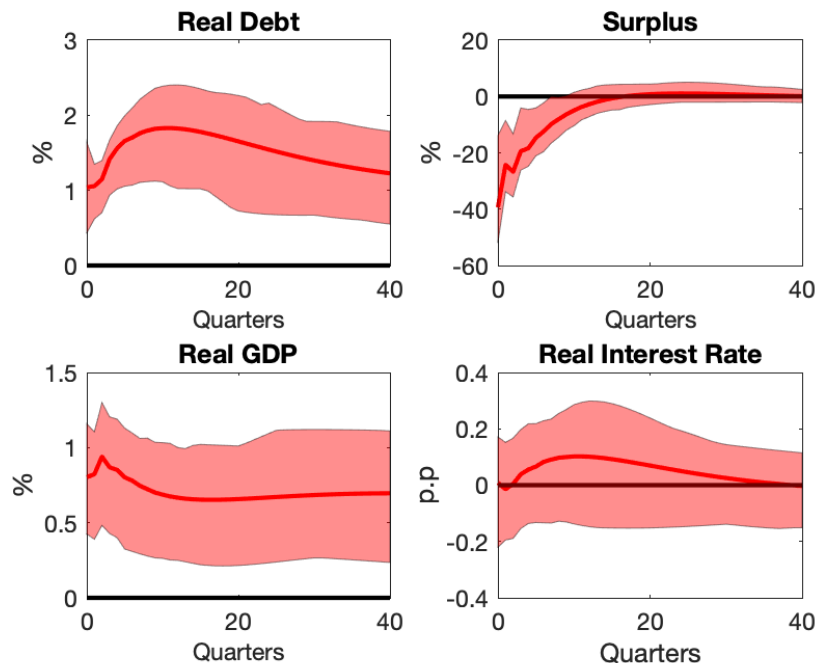
Importantly, we intend not to impose structural interpretations on these innovations in the empirical part of the paper. These innovations generate credit cycles. We focus on the co-movement and the cyclical properties of debt and future surpluses (flows to

creditors) in those credit cycles.

2.4 Empirical Results

Figure 1 plots the responses to a positive max-share innovation to the total liabilities. The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following Kilian (1998)'s method. On average, a rise in debt is associated with an increase in real GDP. This procyclicality of firms' debt is well-documented in the literature, see e.g., Covas and Haan (2011) and Jermann and Quadrini (2012). We replicate these findings using an alternative empirical strategy that is based on a VAR.

Figure 1: Responses to Total Liabilities Innovations (Max-share)



Note: This figure plots the IRFs to a positive max-share innovation to total liabilities. The max-share innovation is constructed as the linear combination of reduced-form residuals that maximizes the fluctuations in total liabilities at the business cycle frequency. The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following Kilian (1998)'s method.

Interestingly, an innovation that drives up real debt is not followed by increases in *future* surplus. In contrast, flows to creditors drop in the short run before converging

back to their initial level. We label this novel observation as Fact 1. Formally, in Figure 1, one should compare the responses of real debt at time t with the surplus' responses at time $t + h$, $\forall h \in N^+$. The observation that the credit boom is *not* associated with future increases in net flows to creditors needs to be emphasized.⁷ Note that the real interest rate changes insignificantly, both economically and statistically.

Robustness checks The baseline findings are robust to: (i) the use of pre-2008 recession sample (Figure A.1); (ii) the use of max-share innovations to the total debts instead of the total liabilities (Figure A.2); and (iii) the use of alternative innovations constructed using zero restrictions (Figure A.3).⁸ Section 2.5 presents additional detailed discussions/robustness checks that address potential statistical issues.

The Fundamental Component In the next, we compute the responses of the proxy of the fundamental component of total liabilities, in the spirit of Galí and Gambetti (2015). Linearize Equation (5) to obtain:

$$\hat{f}_t = \sum_{h=0}^{\infty} \rho^h [(1 - \rho) \tilde{s}_{t+h+1} - \hat{r}_{t+h+1}], \quad (9)$$

See Appendix 2.2.1 for detailed derivations.

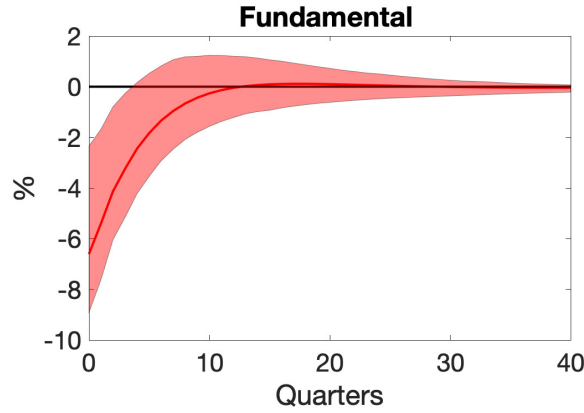
The estimated VAR provides IRFs of $\tilde{s}_{t+h}$ and r_{t+h} . Together with Equation (9), one can compute the IRFs of $\hat{f}_t$. Figure 2 plots the results. The constructed fundamental component decreases significantly in response to a positive innovation to total liabilities—Fact 2. To compute the results displayed in Figure 2, we calibrate ρ to be equal to 0.95. Assigning alternative values to ρ does not alter the result qualitatively: see Figure A.4.

Theoretically, the IRFs of the real interest rate might matter in the construction of the responses of the fundamental component. However, as it can be seen in Figure 1, the real interest rate constructed using the Baa corporate bond yield barely changes, which is not because of the choice of interest rate. Figure A.5 plots the responses of the fundamental component constructed on the basis of alternative VARs that include other measures of real interest rates: the US corporate index effective yield, the BBB US corporate

⁷Quantitatively, surplus drops by 40% (of its sample mean) on impact. In the data, on average, the surplus is approximately 6% of the real debt in absolute value. Therefore, the decrease in surplus is roughly 2.4% of the sample mean of real debt.

⁸More specifically, d_t is ordered first in the VAR, and we impose the square root of the variance-covariance matrix to be a lower-triangle matrix. Figure A.3 plots the IRFs to the innovations to d_t —the only shocks that affect d_t contemporaneously.

Figure 2: Responses of the NPV of Surpluses to Total Liabilities Innovations (Max-share)



Note: This figure plots the IRF of the NPV of surpluses to a positive max-share innovation to total liabilities. This IRF is constructed using the IRFs of surpluses and the interest rate, according to equation (9). The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small-sample bias following Kilian (1998)'s method.

index effective yield, and the CCC & lower US high yield index effective yield. The right-hand-side panels show that the US corporate index effective yield, computed on the basis of corporate bonds of different qualities, reacts to the extent of less than 0.1 percentage points in response to one standard deviation increase in the innovation to total liabilities. The corporate bond yield computed using high-risk bonds—CCC & lower US high yield index effective yield—decreases by 0.5 percentage points on impact. The left-hand-side panels confirm that the responses of the approximated fundamental component are barely affected by the use of alternative interest rates.

2.5 Discussions and Robustness Checks

The key empirical observation is that following an increase in total liabilities, the future aggregate surplus does not increase. The remainder of the paper provides a theory rationalizing the findings. However, one might argue that statistical issues can account for the findings. We address these potential concerns in this section.

Aggregation The aggregate cash flow Equation (7) guides the empirical analysis. We now discuss whether and how firm heterogeneity affects the equation. The detailed discussion can be found in Appendix 2.3, where we derive the aggregate cash flow Equation (7) with heterogeneous firms.

Appendix 2.3 shows that the link between aggregate debt and future aggregate surplus remains unchanged in a simple framework with heterogeneous firms. Therefore, the co-movement between the debt and subsequent flows to creditors observed in the data (Fact 1) remains a puzzle.

In an economy with heterogeneous firms, the aggregate cash flow equation features an aggregate interest rate, a weighted average of firm-level interest rates. The weights of various firms' interest rates depend on the relative sizes of firms' surplus and debt. In robustness checks (discussed above in Figure A.5), we show that the response of the constructed fundamental component (Fact 2) is robust to alternative measures of aggregate interest rates. Nevertheless, future empirical research exploring the flow of funds and interest rates at the firm level can be fruitful.

Moreover, the theoretical model presented in Section 3 features heterogeneous firms with entries and exits/defaults. The model serves as a laboratory for interpreting the empirical findings. Simulation exercises presented in Section 4.1 suggest that the issue of aggregation is no concern.

Small-Sample Bias One might argue that the estimated IRFs based on the VAR are subject to a small-sample bias. This is a typical issue for macroeconomic studies based on postwar time-series data. To address this issue, for all IRFs estimated in the paper, we have employed Kilian (1998)'s bias-corrected technique for the point estimates of the IRFs and their confidence intervals. Kilian (1998)'s approach addresses the bias and skewness of small-sample distributions of the IRFs.

Surplus and Flow to Creditors As mentioned above, in the data, one can measure flow to creditors using S_t or, alternatively, $R_t D_{t-1} - D_t = Int_t + D_{t-1} - D_t$ with Int_t denoting interest payment associated with D_{t-1} . The baseline specification uses the former because the latter requires data on interest payments with the following limitations. First, the constructed interest payments (Int_t) contain imputed and estimated components and are subject to bigger measurement errors. Second, Int_t is only available at an annual frequency. The two time series— S_t and $Int_t + D_{t-1} - D_t$ —have a correlation of 0.9362 at the annual frequency.

The following exercise provides a robustness check using $Int_t + D_{t-1} - D_t$ as a measure of flow to creditors or surplus. Figure A.6 presents the findings: the baseline findings are robust to the use of the alternative measure of surplus.

Equities To address the concern that shocks to total equities are driving the empirical findings, we re-estimate the VAR using the total liabilities, surplus, and the real GDP rescaled by the total equities for non-financial corporate and non-corporate firms. Figure A.7 plots the estimated results: the central message remains unaffected.

Misspecified VAR and a Bigger VAR Another concern is that the VAR could be misspecified, which could lead to biased estimates of IRFs. We address this concern in two ways. First, in the data, we show that our empirical findings are robust to the estimation of a bigger VAR model, which contains the following additional variables: total capital expenditure, personal consumption expenditure, unemployment, and inflation. The estimated results (see Figure A.8) are similar to those obtained in the baseline.

Second, the simulations of the theoretical model presented in Section 4.2 demonstrate that the baseline VAR performs well in recovering the transition dynamics (IRFs) of model.

Can the VAR Capture a Potentially Delayed Increase? One potential criticism is that the data-generating process features a potentially delayed increase of surplus—it decreases in the short run but increases in the medium to the long run—that the VAR might fail to capture.

If the delayed increases only occur *far* in the future, using a VAR model with variables in the log level could be problematic because the response of a variable has to converge to zero (or its pre-shock level) in the long run by construction. Note that the criticism relies on the hypothetical assumption that a delayed increase occurs in the long run.

To address this concern, we estimate the same VAR model but using variables in the first difference, for example, $\Delta \tilde{s}_t$. In the first differenced model, while a shock's effect on $\Delta \tilde{s}_t$ has to die out in the long run, the implied effect on s_t does not need to converge to zero. Figure A.9 plots the estimation results. The baseline findings are robust. When compared with the baseline model, using variables in the first difference has the disadvantage of differencing out a potential cointegration relationship in the VAR: a feature that leads to the super-consistency of the parameter estimates.

Moreover, in Section 4.2, we show that, through the simulations of our quantitative model, the baseline VAR model could capture a delayed increase in surplus if it were to exist in the true data-generating process.

2.6 IRFs to Structural Shocks

The baseline findings rely on reduced form innovations that have no structural interpretations. Moreover, we are *not* arguing that *all* fluctuations in liabilities are associated with declining future surpluses. Our findings suggest that there exist shocks that generate the puzzling negative relationship between current total liabilities and future flow to creditors. The puzzling evidence we document does not require the identification of a structural shock. The structural interpretation of these findings is provided through the lens of the theoretical model provided in Section 3.

Nevertheless, it can be helpful to present evidence in response to *structural shocks*. The following two exercises provide such evidence. The theoretical model in Section 3 argues that exogenous sentiment shocks (unrelated to fundamental shocks) are driving the empirical findings. The following two exercises support this argument by providing IRFs to non-fundamental shocks.

IRFs to EBP Shocks First, we estimate IRFs of the key variables to the excess bond premium (EBP) shock constructed by Gilchrist and Zakrajšek (2012). The EBP captures "deviations in the pricing of corporate bonds relative to the measured default risk of the issuer." That is to say, EBP measures changes in the supply of credit to corporate firms that are orthogonal to firms' fundamentals. Following Gilchrist and Zakrajšek (2012), EBP shocks are identified using short-run restrictions. Specifically, EBP shocks are assumed to affect the excess bond premium and real interest rate contemporaneously but do not affect other economic activities (other variables in the VAR) on impact.

Figure A.10 plots the IRFs to a negative EBP shock that raises credit supply. A negative EBP shock is expansionary—real GDP increases, which is consistent with Gilchrist and Zakrajšek (2012)'s original findings.

The current paper focuses on the effects on firms' liabilities and cash flow activities. Similar to the baseline reduced-form findings, Figure A.10 shows that in response to EBP shocks, increases in D_t are not followed by increases in S_t .

IRFs to Non-fundamental Shocks in the Credit Market The following empirical analysis complements the previous exercise by providing an alternative identification strategy to identify non-fundamental shocks in the credit market in the spirit of Tang and Zhang (2022).

Our identification strategy relies on the following assumption. Apart from shocks

to the real interest rate, among the remaining structural shocks in the economy, non-fundamental shocks in the credit market are the ones that maximize the forecast error variance decomposition of the liability-fundamental differential in the subsequent periods.⁹

This identification strategy differs from the innovations constructed in the baseline in an important dimension. Many fundamental shocks, such as productivity and shocks to firms' collaterals, can drive fluctuations in total liabilities through their effects on the *fundamental component* of total liabilities. The identification assumption here is that non-fundamental shocks in the credit market form the primary source of fluctuations in the *liability-fundamental differential component*.

Figure A.11 reports the IRFs to a positive non-fundamental shock. Non-fundamental shocks produce the puzzling negative relationship between current total liabilities and future flows to creditors.

In the remainder of the paper, we argue that sentiment shocks, specifically, changes in the D_t^∞ component, can explain these empirical findings.

3 Model

Our model describes a dynamic economy with heterogeneous firms. Firms have access to a frictional credit market, where they issue defaultable debt. The model extends canonical firm dynamics models with financial frictions,¹⁰ by relaxing the no-Ponzi-game condition for firm optimization problem, i.e., credit bubbles exist.¹¹

3.1 Households

Time is discrete and infinite. There is no aggregate uncertainty. The economy is populated by overlapping generations of risk-neutral households who live indefinitely. Households

⁹We apply the identification strategy to our VAR with r_t being first in the order. The innovations to r_t constructed using the zero restrictions are called shocks to the real interest rate. The non-fundamental shocks are the linear combinations of the remaining reduced-form residuals that maximizes the fluctuations in $\hat{d}_t - \hat{f}_t$ at the business cycle frequency.

¹⁰See for example Khan and Thomas (2013), Arellano et al. (2019), Ottonello and Winberry (2020).

¹¹We model credit bubbles in the spirit of Tirole (1985). See also Domeij and Ellingsen (2018), Farhi and Tirole (2012), Ikeda and Phan (2019), and Martin and Ventura (2012).

maximize lifetime utility

$$U_{it} = E_t \sum_{\tau=0}^{\infty} \beta^{\tau} C_{it+\tau} \quad (10)$$

where $\beta \in (0,1)$ is the discount factor, and $C_{it} > 0$ denotes the consumption of goods.¹²

The size of a given cohort remains fixed over time, and is $1 + g$ of the cohort size of its predecessor, i.e., the size of the new cohort M_t follows

$$M_{t+1} = (1 + g) M_t. \quad (11)$$

Every period the population is joined by a cohort of increasing size; consequently, the total population size grows at g .

3.2 Firms

Firms use capital to produce a homogeneous product. The product is sold in a frictionless market. The price of the product is normalized to one. The production technology displays decreasing returns to scale

$$y_{jt} = A_t \varphi_{jt} k_{jt}^{\alpha}, \quad (12)$$

with $\alpha \in (0,1)$. A_t denotes a common productivity component, identical across all firms. φ_{jt} represents idiosyncratic productivity independent and identically distributed (i.i.d) across different firms. k_{jt} denotes predetermined capital stock, of which δk_{jt} is depreciated in the process of production. For any existing firm j , the idiosyncratic productivity φ_{jt} follows a Markov process

$$\log \varphi_{jt+1} = \rho \log \varphi_{jt} + \varepsilon_{jt+1}, \quad (13)$$

where $\rho \in (0,1)$, $\varepsilon_{jt} \sim N(0, \sigma^2) \forall t, \forall j$. Production incurs a fixed operation cost c^f . The profit of operating a firm j , before interest payment, is given by

$$\pi_{jt} \equiv y_{jt} - c^f. \quad (14)$$

Every period, the newborn households have the opportunity to create new firms that

¹²The setup is similar to a typical perpetual youth model, as developed by [Blanchard \(1985\)](#), who introduces a constant probability of death of individuals. In our model, individuals survive forever, and thus the model can be viewed as a special case of perpetual youth model with zero death probability. See also [Olivier \(2000\)](#), [Galí \(2020\)](#), and [Tang and Zhang \(2022\)](#) for examples of perpetual youth and rational bubbles.

start producing in the subsequent period. Each newborn individual has access to a unit continuum of blueprints, which have the potential to become firms. We name these blueprints potential entrants. Each potential entrant draws its idiosyncratic productivity from the distribution

$$\varphi_{jt} \sim \log N \left(-m \frac{\sigma}{\sqrt{1-\rho^2}}, n \frac{\sigma^2}{1-\rho^2} \right),$$

where m and n capture the possible difference between the initial productivity draw and the idiosyncratic productivity process.¹³ We label potential entrants who choose to enter the market as *entrants*.

3.3 Creditors

Firms can issue one-period state-noncontingent debts but cannot issue new equity. Firms' debts are purchased by a representative, perfectly competitive financial intermediary that receives deposits from households. A firm j that promises to repay b_{jt+1} at $t+1$ receives $q_{jt}b_{jt+1}$ at t . The following breakeven condition pins down the average debt price schedule q_{jt}

$$q_{jt}b_{jt+1} = \frac{1}{R_{t+1}} \left[\left(p_{jt+1}^n b_{jt+1} + \left(1 - p_{jt+1}^n \right) \gamma_{t+1} k_{jt+1} \right) \right], \quad (15)$$

where R_{t+1} is the interest rate that creditors pay to households, γ_{t+1} denotes the pledgeability of capital stock k_{jt+1} , and p_{jt+1}^n denotes the probability of repaying. If a firm defaults, it repays to the extent of the collateral assets $\gamma_t k_{jt}$ to its creditors and *exits* the market. Note that p_{jt+1}^n is an endogenous variable. We postpone the discussion of firms' default decisions in Section 3.4.

The dividend of a surviving firm is given by

$$div_{jt} = \pi_{jt} - b_{jt} - I_{jt} + q_{jt}b_{jt+1}. \quad (16)$$

Rearrange Equation (16)

$$b_{jt} = \pi_{jt} - div_{jt} - I_{jt} + q_{jt}b_{jt+1}.$$

¹³Potential entrants do not produce until they become firms in the subsequent period. Hence, the initial draw can be interpreted as a signal of future productivity upon entering the market.

Define the net flow to creditors (f_{jt}) as

$$f_{jt} \equiv \begin{cases} \pi_{jt} - \text{div}_{jt} - I_{jt}, & \text{with prob. } p_{jt}^n \text{ (if repay)} \\ \gamma_t k_{jt}, & \text{with prob. } 1 - p_{jt}^n \text{ (if default)} \end{cases}.$$

The breakeven condition (15) can be rewritten into

$$q_{jt} b_{jt+1} = \frac{1}{R_{t+1}} E_t \{ f_{jt+1} + q_{jt+1} b_{jt+2} \}. \quad (17)$$

Note that in (17), $q_{jt+1} b_{jt+2} = 0$ if firm j defaults at $t + 1$, since firm j exits the market after defaulting. Solving (17) forward leads to

$$q_{jt} b_{jt+1} = \sum_{\tau=t}^{\infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t f_{j\tau+1} + \lim_{\tau \rightarrow \infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t q_{j\tau} b_{j\tau+1}, \quad (18)$$

which corresponds to the debt decomposition aforementioned in Section 2.1.

No-Ponzi-game Condition In the literature, it is conventionally imposed that, for all t , firms are subject to

$$\lim_{\tau \rightarrow \infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t q_{j\tau} b_{j\tau+1} = 0. \quad (19)$$

Equation (19) is typically referred to as the no-Ponzi-game condition. (17) and (19) imply that

$$q_{jt} b_{jt+1} = \sum_{\tau=t}^{\infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t f_{j\tau+1}.$$

As long as the no-Ponzi-game condition (19) holds, the income of debt issuance is equal to the NPV of future net flows to creditors.

Credit Bubbles We relax the no-Ponzi-game condition (19). Particularly, we allow b_{jt} to be composed of two types of debt

$$b_{jt} = F_{jt} + B_{jt}.$$

F_{jt} denotes secured debt from which the lenders can retrieve $\gamma_{t+1}k_{jt+1}$ if firm j defaults at time $t + 1$. The income from issuing F_{jt+1} is equal to

$$q_{jt}^F F_{jt+1} = \frac{1}{R_{t+1}} \left[\left(p_{jt+1}^n F_{jt+1} + \left(1 - p_{jt+1}^n \right) \gamma_{t+1} k_{jt+1} \right) \right], \quad (20)$$

where q_{jt}^F denotes the price of F_{jt+1} . Besides, the issuance of F_{jt+1} is subject to the no-Ponzi-game condition, i.e.,

$$\lim_{\tau \rightarrow \infty} \left(\prod_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t q_{j\tau}^F F_{j\tau+1} = 0. \quad (21)$$

In addition, B_{jt} denotes unsecured debt, from which the lenders cannot retrieve any collateral upon default. The income from the issuance is equal to

$$q_{jt}^B B_{jt+1} = \frac{1}{R_{t+1}} p_{jt+1}^n B_{jt+1}, \quad (22)$$

where q_{jt}^B denotes the price of B_{jt+1} . As opposed to F_{jt+1} , the issuance of B_{jt+1} is not subject to the no-Ponzi-game condition. Instead, we impose an alternative restriction: for any incumbent firm j

$$q_{jt}^B B_{jt+1} \leq B_{jt}. \quad (23)$$

(23) implies that incumbent firm j can fully roll over its existing unsecured debt B_{jt} , but not issue new extra unsecured debt. In addition, we assume that potential entrants can issue

$$q_{jt}^B B_{jt+1} \leq s_{jt} B_0, \quad (24)$$

where $s_{jt} \in \{0, 1\}$ is i.i.d. across potential entrants. $s_{jt} = 1$ with the probability p_B . $1 - p_B$ of potential entrants are subject to no-Ponzi-game condition and can only issue the secured debt F_{jt+1} .

Note that the unsecured debt described above is analogous to the "unbacked public debt" by [Hellwig and Lorenzoni \(2009\)](#). We discuss the relevance and interpretations of constraints (23) and (24) in Section 3.6.

In a nutshell, firms can only issue a secured debt F_{jt} , that cannot be indefinitely rolled over, and an unsecured debt B_{jt} , that can be indefinitely rolled over. Section 3.6 generalizes this assumption and shows that the findings are unaffected.

3.4 Timing

Each period t consists of multiple stages for a firm to take action. At the beginning of a period, all shocks are realized. Firms then choose whether to default on their outstanding debts. If they default, they lose all the capital stock and exit the market immediately before production takes place. A fraction of the capital stock, $\gamma_t k_{jt}$, is retrieved and repaid to their creditors. The rest of the capital stock is lost in the liquidation process. Firms that decide not to default produce according to the production function (12).

After production, firms (i) repay their debt to the creditors, (ii) choose the dividends for their shareholders, and (iii) decide how much to invest for future production. The investment is funded by internal resources and external finance via one-period state-noncontingent debts. We impose a standard simplifying assumption in macro-finance literature: firms are prohibited from financing through equity. In other words, they cannot pay negative dividends to their shareholders. Meanwhile, entrants enter the economy and make investments. All entrants start with zero capital stock and can only borrow through the frictional debt market.

It is important to note that firms only default if paying negative dividends is inevitable. The value of default is zero; thus, firms only *voluntarily* default if the value of repaying debt is negative. However, the value of repaying debt is equal to the sum of current and discounted future dividends, which are always non-negative. Therefore firms only default if production inevitably leads to negative dividends.

The optimization problems of existing firms and potential entrants are formalized in Section 3.5.

3.5 Equilibrium Analysis

Firms' Optimization Problem Define net worth x_{jt} as

$$x_{jt} \equiv \pi_{jt} + (1 - \delta)k_{jt} - F_{jt} - B_{jt}.$$

Net worth is equal to the profit of operating net of the debt repayment (b_{jt} , that is, the sum of F_{jt} and B_{jt}) in addition to the un-depreciated capital. Hereafter, we suppress the subscripts of variables in functions to minimize the notational burdens. For notation ease, we use $\lambda \equiv \{B, x\}$ to denote idiosyncratic state variables other than idiosyncratic

productivity φ . The value function of an incumbent firm is given by

$$V_t(\varphi, \lambda) = \max_{F', B', k'} \left\{ x - k' + q_t^F(\varphi, F', B', k') F' + q_t^B(\varphi, F', B', k') B' + \frac{1}{R_{t+1}} \int V_{t+1}(\varphi', \lambda') dJ(\varphi'|\varphi) \right\}, \quad (25)$$

$$x - k' + q_t^F(\varphi, F', B', k') F' + q_t^B(\varphi, F', B', k') B' \geq 0, \quad (26)$$

$$q_t^B(\varphi, F', B', k') B' \leq B, \quad (27)$$

where $V_t(\varphi, \lambda)$ denotes the value function of an incumbent firm, and $J(\varphi'|\varphi)$ represents the transition probability of φ .¹⁴ The choice of control variables is subject to (26), the non-negative-dividend constraint, and (27), the limit on B' —the debt that can be rolled over indefinitely. Note that every firm uses $\frac{1}{R_{t+1}}$ as the *discount factor* for future dividends. First, every single firm generates the same expected return, otherwise the market for shares cannot be cleared as *risk-neutral* investors only purchase shares with the highest expected return. Second, the expected return on shares is equal to the interest rate paid by the financial intermediary on deposits. On the one hand, if the expected return on shares is higher than the interest rate, there is no lending and the credit market cannot be cleared. On the other hand, if the expected return on shares is less than the interest rate, the market for shares cannot be cleared as investors have no incentive to hold shares.¹⁵

Similarly, the value function of a firm entry is equal to

$$V_t^e(\varphi, s) = \max_{F', B', k'} \left\{ -k' + q_t^F(\varphi, F', B', k') F' + q_t^B(\varphi, F', B', k') B' + \frac{1}{R_{t+1}} \int V_{t+1}(\varphi', \lambda') dJ(\varphi'|\varphi) \right\}, \quad (28)$$

$$-k' + q_t^F(\varphi, F', B', k') F' + q_t^B(\varphi, F', B', k') B' \geq 0, \quad (29)$$

$$q_t^B(\varphi, F', B', k') B' \leq sB_0. \quad (30)$$

where $V_t^e(\varphi, s)$ denotes the value function of an entrant. $s \in \{0, 1\}$ determines whether a firm receives indefinitely rollover-able credit: $s = 1$ implies that the firm can issue an indefinitely rollover-able debt B_0 , otherwise ($s = 0$) the firm's total debt is subject to no-Ponzi-game condition. A potential entrant enters the market if and only if

$$V_t^e(\varphi, s) \geq 0.$$

¹⁴This paper assumes that there is no equity price bubble. See Tang and Zhang (2022) for a model of equity price bubbles and firm dynamics.

¹⁵Throughout this paper, we assume a unique solution to policy functions and value functions exists. Following the typical procedure in the literature, we verify that our iterative numerical solution leads to the same results, given different initial guesses. We also assume that the probability of repayment is greater than zero over the state space, verifiable in the numerical exercise.

Proposition 1. $\forall t$, it is optimal for an incumbent firm j to choose

$$q_{jt}^B B_{jt+1} = B_{jt}, \quad (31)$$

and for an entrant j to choose

$$q_{jt}^B B_{jt+1} = s_{jt} B_0. \quad (32)$$

Proof. See Appendix 3.1. □

The intuition for Proposition 2 is the following. Firms can persistently roll over a fraction of their debt, B , without using their own resources (f in Equation (18)) to repay B . Therefore, it is always optimal to maximize borrowing this type of debt. Henceforth we impose (31) and (32) in the following analysis.

Note that there exists no upper bound for B_{jt} as firms permanently roll over these debts. However, in equilibrium, the aggregate credit/GDP ratio should be limited. Otherwise, resources would be depleted, and rational expectations can rule out the equilibrium (Tirole 1985). In Section 3.6, we present the existence condition for a sustainable aggregate credit bubble along a balanced growth path.

Equation (27) implies that incumbents can only roll over the existing B but cannot issue an extra B . Without constraints (27) and (30), firms would borrow unlimited amount of B . The existing macroeconomic models typically impose the no-Ponzi-game condition so that the optimization problem is well defined. We impose (27) and (30) as an alternative for the no-Ponzi-game condition.

The debt decomposition Equation (18) can be rewritten into

$$q_{jt}^F F_{jt+1} + q_{jt}^B B_{jt+1} = \sum_{\tau=t}^{\infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t f_{j\tau+1} + \lim_{\tau \rightarrow \infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t \left(q_{j\tau}^F F_{j\tau+1} + q_{j\tau}^B B_{j\tau+1} \right). \quad (33)$$

According to Equations (22) and (31)

$$q_{jt}^B B_{jt+1} = \frac{1}{R_{t+1}} E_t B_{jt+1} = \frac{1}{R_{t+1}} E_t q_{jt+1}^B B_{jt+2}. \quad (34)$$

Solving (34) forward

$$q_{jt}^B B_{jt+1} = \lim_{\tau \rightarrow \infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t q_{j\tau}^B B_{j\tau+1}. \quad (35)$$

Combining (33) and (35), we get

$$q_{jt}^F F_{jt+1} = \sum_{\tau=t}^{\infty} \left(\prod_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t f_{j\tau+1}. \quad (36)$$

Equation (36) implies that $q_{jt}^F F_{jt+1}$ is equal to the NPV of future net flows to creditors, which we label as fundamental credit, or the fundamental component. Equation (35) demonstrates that $q_{jt}^B B_{jt+1}$ is not backed by future flows to creditors, but instead by future credits. We label $q_{jt}^B B_{jt+1}$ as the bubble component.

Reformulating Firms' Optimization Problem Define adjusted net worth

$$\tilde{x}_{jt} \equiv x_{jt} + q_{jt}^B B_{jt+1} = \pi_{jt} + (1 - \delta) k_{jt} - F_{jt}. \quad (37)$$

Proposition 2 enables us to reformulate the optimization problems. First, the idiosyncratic states can be fully summarized by $\varphi, \tilde{x}$. Therefore we can replace λ with $\tilde{x}$. Second, since B' can always be rolled over, it does not affect the expected dividend upon production and the probability of default. Thus we can omit B' in the price schedule function. Henceforth we redefine the price schedule function by $q_t^F(\varphi, F', k')$, removing B' from the original debt price schedule function $q_t^F(\varphi, F', B', k')$ in (25)-(27), and (28)-(30).

In a nutshell, (25)-(27) can be rewritten into

$$V_t(\varphi, \tilde{x}) = \max_{F', k'} \left\{ \tilde{x} - k' + q_t^F(\varphi, F', k') F' + \frac{1}{R_{t+1}} \int V_{t+1}(\varphi', \tilde{x}') dJ(\varphi' | \varphi) \right\}, \quad (38)$$

$$\tilde{x} - k' + q_t^F(\varphi, F', k') F' \geq 0. \quad (39)$$

(28)-(30) can be rewritten into

$$V_t^e(\varphi, s) = V_t(\varphi, sB_0). \quad (40)$$

(38) and (39) describe a typical financially-constrained optimization problem in the literature.¹⁶ We follow a standard numerical method to solve for the value functions and policy functions. The details are presented in Appendix 4.

¹⁶For example, see Ottonello and Winberry (2020).

Dynamic Equilibrium A dynamic equilibrium should include value functions, policy functions, prices, and the measure of firms, such that (i) firms and households optimize, (ii) the law of motion of firms' distribution is consistent with firms' decision-making, and (iii) all markets are cleared. We define the equilibrium of our model in Appendix 3.2.

Our analysis focuses on a balanced growth path (henceforth BGP) and transition dynamics around it. Along the BGP, the relative distribution of firms remains unchanged, but the number increases at g . The interest rate R_t is equal to $\frac{1}{\beta}$ along a BGP: if $R_t > \frac{1}{\beta}$, no one ever consumes, and the transversality condition is violated; if $R_t < \frac{1}{\beta}$, the credit and equity market cannot be cleared as no one invests.

3.6 Discussions

Appendix 3.5 presents a stylized example to help develop intuition about credit bubbles and to provide a better understanding of the purposes of the model assumptions and the main results in our full model.

Below, we discuss selected features and assumptions of the model.

Self-enforcing Debt Unsecured debts (credit bubbles) described by (23) and (24) are analogous to "self-enforcing private debts" and "unbacked public debts" by Hellwig and Lorenzoni (2009). As implied by (34), $\forall j, t$

$$q_{jt}^B B_{jt+1} = \frac{1}{R_{t+1}} E_t B_{jt+1} = \frac{1}{R_{t+1}} E_t q_{jt+1}^B B_{jt+2},$$

firms have *no* incentive to default on unsecured debts as the debts are perpetually repaid by future debt from lenders. The right-hand-side of (34) can be viewed as *collateral*, even though the collateral is composed of future debts rather than assets or future revenues.¹⁷ Defaulting leads to the loss of this collateral, meaning that firms lose access to the future debts $\frac{1}{R_{t+1}} E_t q_{jt+1}^B B_{jt+2}$.

(34) has three major implications. First, in our setup, firms' unsecured debt reaches the *limit* that can be covered by future indefinite rollover, which is represented by $\frac{1}{R_{t+1}} E_t q_{jt+1}^B B_{jt+2}$. If a firm borrows more than this limit, the debtor would need to commit future net outflows to creditors, and the additional debt (that exceeds the limit) would no longer be

¹⁷The concept of collateral here is analogous to the notion of "bubbly collateral" proposed by Martin and Ventura (2016).

equivalent to a credit bubble.¹⁸

Second, as the unsecured debts are, in fact, fully *secured* by future rollover, taking these debts does not incur any future net outflows. Therefore taking these debts does not affect firms' default probability and borrowing cost, given idiosyncratic productivity φ and decisions F' and k' .

Third, since a firm can issue unsecured debts without raising its borrowing cost, if it were allowed to issue its own preferred level of unsecured debts, it would choose to issue an infinite amount of unsecured debt (and pay infinite dividends) and roll over the debts perpetually. This is why in our model (and in many other models of rational bubbles), firms are not allowed to issue an unconstrained level of unsecured debts (credit bubbles); instead, the unsecured debt are limited by the bubbly collateral, which is determined by sentiment, or market expectation of future debts.¹⁹

Debt Decomposition Until now, we have studied a scenario where indefinite rollover arises only in unsecured debts. The drastic assumption is for clarification purposes. In Appendix 3.3, we demonstrate that F_{jt} and B_{jt} are respectively equivalent to *fundamental component* and *non-fundamental/bubble component* of total debt b_{jt} , in a more generic scenario.

Specifically, Appendix 3.3 demonstrates that provided a firm can indefinitely roll over a fraction of its debts, regardless of whether these debts are secured by collateral, it solves the optimization problem as if it would issue only the two types of debts, F and B , which are defined in Section 3.3.

In a nutshell, although in reality firms may not necessarily issue the specific unsecured debt B as described in Section 3.3, as long as their debts contain a bubble component, one can view the credit bubble component as the unsecured debts.

The Sustainability of Bubbles As firms can permanently roll over a fraction of their debts, no upper bound exists for the firm-level credit/output ratio. However, in equilibrium, the aggregate credit/GDP ratio should be limited. Otherwise, the resources in the economy are insufficient to sustain the credit market. In Appendix 3.4, we demonstrate that indefinite rollovers are sustainable on a BGP only if $\beta(1 + g) > 1$, i.e., $R < 1 + g$.

¹⁸In their study of unsecured debt, Azariadis et al. (2016) note the similarity between unsecured debts and bubbles as the size of unsecured debts depends on market sentiment. However, unlike our indefinite rollover component, unsecured debts in Azariadis et al. (2016) are backed by expected future net flows to creditors; therefore, those debts belong to the fundamental components.

¹⁹See also Hellwig and Lorenzoni (2009) and Martin and Ventura (2016).

Table 1: Fixed Parameters

Parameter	Description	Value
α	Decreasing returns to scale	0.65
δ	Depreciation rate	0.10
β	Discount factor	0.98
g	Growth rate	3.2%
A	Common TFP component	1
γ	Pledgeability of capital stock	0.916
ρ	Idiosy. shock persistence	0.70
σ	Idiosy. shock volatility	0.38

Note: This table reports fixed parameters.

4 Results

Calibration We calibrate the model along a BGP at an annual frequency. Table 1 lists the fixed parameter values. We choose the decreasing returns to scale $\alpha = 0.65$, and the capital depreciation rate $\delta = 0.1$. We set the discount factor $\beta = 0.98$, implying that $R = 1.02$. The growth rate is set to $g = 3.2\%$, corresponding to the average growth rate of real GDP in the U.S. between 1948 and 2019. Using the estimate by [Hennessy and Whited \(2007\)](#), we set the pledgeability $\gamma = 0.916$. Without loss of generality, we set common TFP component $A = 1$. Finally, regarding idiosyncratic productivity shocks, we choose $\rho = 0.7$ and $\sigma = 0.38$ to match the estimates by [İmrohoroglu and Tüzel \(2014\)](#).

Table 2 presents our calibration of the remaining parameters. We target three types of moments. First, we target moments regarding business dynamism. We use the data from Business Dynamics Statistics (BDS) to compute the share of new establishments, the exit rate among new establishments, and the share of establishments at age 4. Second, we target the investment behavior of firms: the mean investment rate as reported by [Crouzet and Mehrotra \(2020\)](#), who use the extensive micro data underlying the Quarterly Financial Report by the US Census Bureau. Finally, we target firms' financial characteristics. We compute the mean spread, measured by the yield on BAA-rated corporate bonds relative to a 10-year treasury bond, and the mean leverage ratio, as reported by [Crouzet and Mehrotra \(2020\)](#). Panel A lists the estimated parameter values that minimize the difference between the targets and their model-implied counterparts. Panel B reports the target values and model fitness.

Overall, our model fits most of the selected moments well. The model tends to over-

Table 2: Calibration and Model Fit

Panel A: Calibrated Parameters		
Description	Parameter	Value
Mean shift of entrants' prod.	m	0.215
Std shift of entrants' prod.	n	0.014
Initial bubble component	B_0	207.4
Fixed cost of production	c_f	2.831
Fraction of bubbly potential entrants	p_B	0.038

Panel B: Targets and Model Fit		
Moment	Data	Model
Share of new establishments	0.087	0.061
Share of four-year-old establishments	0.045	0.030
Exit rate of new establishments	0.244	0.200
Mean investment rate	0.263	0.553
Mean spread	0.024	0.013
Mean leverage ratio	0.344	0.344

Note: This table reports the calibrated parameters, the moments used in the calibration, and the model fit.

predict the mean investment rate and underpredict the mean spread. One potential explanation is the lack of adjustment cost in our model.²⁰ Hence, in the model, young firms can quickly accumulate capital stock and lower the borrowing cost, leading to a high mean investment rate and a low mean spread relative to the data.

4.1 Impulse Responses

The structural model can be used as a laboratory to assess the effects of various shocks. We compute the transition dynamics following unexpected shocks. The shocks are realized at $t = 0$, before which the economy is on its BGP. We assume the economy follows a perfect foresight equilibrium path from $t = 1$ and returns to BGP at $t = 100$.

Rollover Shocks We follow [Martin and Ventura \(2016\)](#) and consider a shock to credit bubble components. According to Equation (31), in the absence of aggregate shocks, incumbent firm j is supposed to receive $q_{jt}^B B_{jt+1} = B_{jt}$ to renew the existing B_{jt} at t . We

²⁰Incorporating capital adjustment cost would introduce an additional state variable k to the value function maximization problems studied in Section 3.5, albeit not qualitatively affecting the mechanism we detail below to rationalize the dynamics of credit fundamental component.

assume that at $t = 0$, firms receive a shock at their borrowing constraint (23):

$$q_{jt}^B B_{jt+1} \leq (1 + \epsilon^B) B_{jt}, \quad (41)$$

where $\epsilon^B > 0$ represents a surprise increase in the bubble component. Similar to (31) and (32), it is optimal for firms to set $q_{jt}^B B_{jt+1} = (1 + \epsilon^B) B_{jt}$.²¹ Along the perfect foresight equilibrium path afterward, incumbent firms can only roll over the existing indefinite rollover component, i.e., $q_{jt+h-1}^B B_{jt+h} = B_{jt+h-1}$, $\forall h > 1$.

As in Martin and Ventura (2016), such a shock can be interpreted as a change in market sentiment or expectation. The newly issued $\epsilon^B B_{jt}$, together with the existing B_{jt} , are rolled over period by period after t . According to (34),

$$q_{jt}^B B_{jt+1} = \frac{1}{R_{t+1}} E_t q_{jt+1}^B B_{jt+2} = \lim_{\tau \rightarrow \infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t q_{j\tau}^B B_{j\tau+1}.$$

Hence a surprise increase in $q_{jt}^B B_{jt+1}$ is driven by a change in the market expectation of future credit bubbles. As aforementioned, the RHS of (34), $E_t q_{jt+1}^B B_{jt+2}$, can be viewed as the collateral for credit bubbles. The rollover shock raises the debt capacity for credit bubbles by increasing the value of "bubbly collateral", i.e., the expectation of future credit bubbles.

Such a shock is equivalent to an unexpected increase in firms' adjusted net worth. For incumbent firms

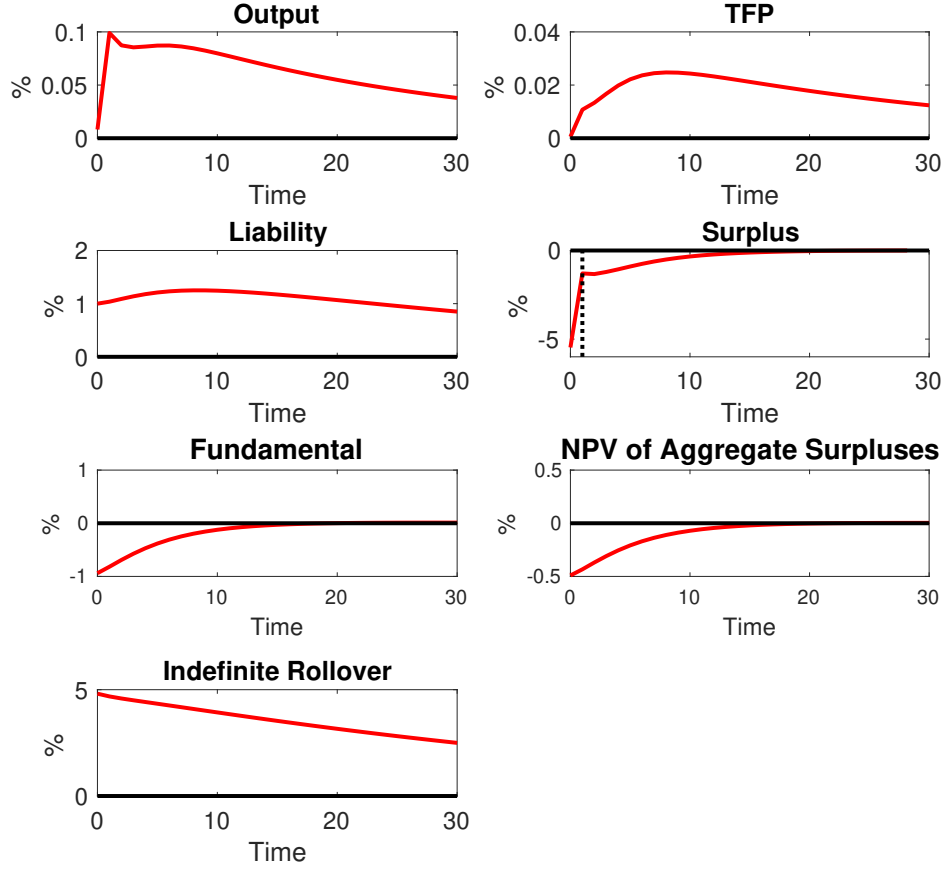
$$\tilde{x}_{jt} \equiv x_{jt} + q_{jt}^B B_{jt+1} = \pi_{jt} + (1 - \delta) k_{jt} - F_{jt} + q_{jt}^B B_{jt+1} - B_{jt},$$

where k_{jt} , F_{jt} , and B_{jt} are predetermined, and π_{jt} is determined by φ_{jt} and k_{jt} . In the absence of rollover shocks, $q_{jt}^B B_{jt+1} = B_{jt}$. As to the incumbents with $B_{jt} > 0$, the rollover shock at $t = 0$ induces $q_{jt}^B B_{jt+1} - B_{jt} = \epsilon^B B_{jt} > 0$, which implies an unexpected increase by $\epsilon^B B_{jt}$ in $\tilde{x}_{jt}$.

Figure 3 plots the impulse responses to the rollover shock. The positive rollover shock allows firms to borrow more (using credit bubbles) and thus stimulates aggregate output, TFP, and credit. Relative to the output increase, the credit expansion is disproportionately large. Contrary to conventional wisdom, the credit boom coincides with a persistent drop in the net flow to creditors (surplus) and fundamental credit. Despite the fundamental credit declines following the rollover shock, the total credit increases due to the rise in

²¹See Appendix 3.1.

Figure 3: Model Prediction: Responses to Indefinite Rollover Shocks



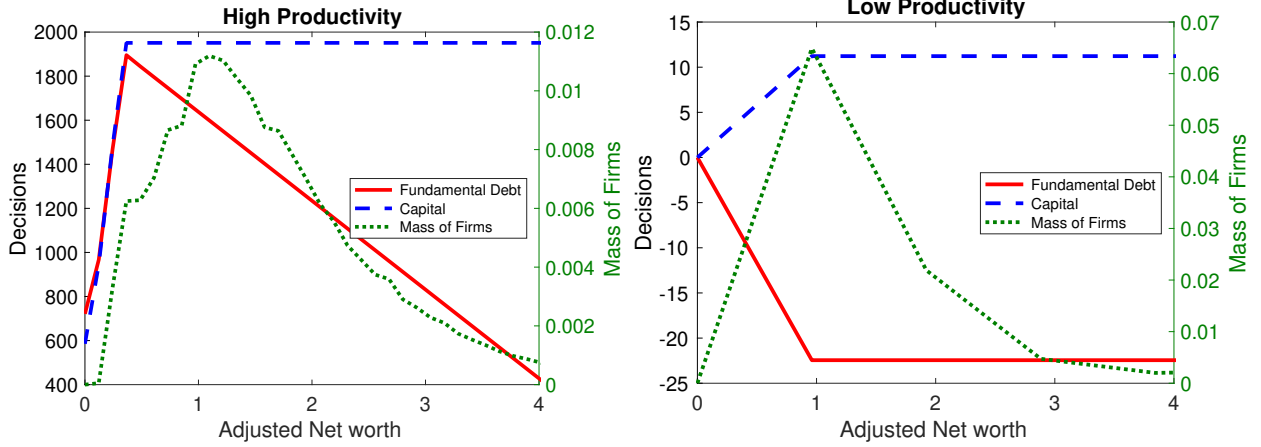
Note: This figure plots the IRFs to indefinite rollover shocks in the model. The shock is normalized to increase the aggregate liability on impact by 1%.

credit bubbles.

Additionally, we plot the NPV of the aggregate surpluses, as in Section 2.4. Note that the NPV of the aggregate surpluses is not necessarily equal to the aggregate fundamental credit. Figure 3 shows that the impulse responses of the NPV of the aggregate surpluses are qualitatively analogous to the impulse responses of the aggregate fundamental credit.

Understanding the Mechanism We now discuss the mechanism. As aforementioned, the rollover shock is equivalent to an increase in adjusted net worth. Figure 4 plots the policy functions of firms along the BGP, as functions of adjusted net worth $\tilde{x}$. The left-hand-side panel considers firms with relatively high productivity. Capital investment k' is increasing in $\tilde{x}$, whereas fundamental credit $q_{jt}^F F_{jt+1}$ is non-monotone in $\tilde{x}$. As to firms with low $\tilde{x}$, collateral assets are scarce, and the increase in $\tilde{x}$ helps boost fundamental

Figure 4: Model Mechanism: Policy Functions



Note: This figure plots the policy functions. Specifically, the figure plots the fundamental debt (solid red line) and capital (dashed blue line) as functions of the adjusted net worth. The dotted green line plots the distribution of firms. The left (right) panel plots the policy functions for firms with $\log \varphi$ 1.25 standard deviation above (below) the unconditional mean of the idiosyncratic productivity process. The unit on the horizontal axis is the standard deviation of the adjusted net worth of all firms. The 0 points on the horizontal axis represent the cutoff adjusted net worth levels below which firms default, given different productivity levels.

credit to increase investment further. On the contrary, firms with high $\tilde{x}$ likely have accumulated abundant capital stock. These firms are prone to insure against idiosyncratic risk by saving the newly gained adjusted net worth to lower fundamental credit. The right-hand-side panel considers firms with relatively low productivity, whose desired level of capital of which is relatively low. The precautionary channel dominates, and therefore, the fundamental credit $q_{jt}^F F_{jt+1}$ is monotonically decreasing in $\tilde{x}$. In summary, the shock has heterogeneous effects across firms. The overall effects of the rollover shock depend on the distribution of φ , $\tilde{x}$, and B across firms.

The precautionary channel highlighted above is novel to the literature. First, as other firm dynamics models with financial frictions, our model features uninsurable idiosyncratic risk, which gives rise to precautionary saving. Second, similar to [Hellwig and Lorenzoni \(2009\)](#) and [Martin and Ventura \(2016\)](#), our model considers future debts as collateral. On the one hand, credit bubbles are fully secured by future credit bubbles. Given idiosyncratic productivity φ and decisions k' and F' , increasing credit bubbles does not raise firms' default probability. On the other hand, fundamental credit F' entails future net outflows to creditors and increases default risk. Therefore, firms can substitute fundamental credit with credit bubbles to reduce the future probability of default if they

are allowed to issue more credit bubbles. Hence, in our model, firms can *borrow more* to become safer. This contrasts a standard model of precautionary saving, where agents typically need to save more (or borrow less) to increase their safety.

Finally, note that precautionary saving arises in risk-neutral firms since non-insurable idiosyncratic shocks may increase default probability and, consequently, the borrowing cost. Hence financially *constrained* firms are incentivized to insure against idiosyncratic risks through accumulating financial wealth or reducing fundamental credit.²²

Other Financial Shocks Rollover shocks belong to the broader class of financial shocks that change firms' financing conditions and borrowing capacity. We compare the rollover shock with a standard reduced-form financial shock: a shock to pledgeability γ . We assume γ increases unexpectedly by ϵ at $t = 0$, and the increment decays at the rate $\rho_\gamma = 0.8$ per period. At $t = 100$, γ reverts to the BGP level.

Figure A.12 plots the impulse responses to the pledgeability shock. The pledgeability shock boosts the aggregate output, TFP, and credit. Overall, the pledgeability shock and the rollover shock have qualitatively similar aggregate implications. However, unlike the rollover shock, the pledgeability shock also raises the future net flows to creditors (following a short-run decline) and the fundamental credit. The increase in fundamental credit drives the increase in total credit entirely. These results are qualitatively consistent with the results obtained in the literature.

In a nutshell, incorporating credit bubbles provides a novel interpretation of financial shocks that give rise to credit cycles, highlighted by, e.g., [Jermann and Quadrini \(2012\)](#) and [Khan and Thomas \(2013\)](#). More importantly, shocks to credit bubbles reconcile our empirical findings that credit booms coincide with a declining fundamental component and are followed by declining net flows to creditors.

Productivity and News Shocks We now turn to the effects of real shocks. First, consider a shock to the common TFP component A_t . At $t = 0$, A_t increases by ϵ_A . The innovation falls at the annual rate $\rho_A = 0.8$ from $t = 1$ to $t = 99$ and reverts to zero at $t = 100$. The impulse responses of the productivity shock are displayed in Figure A.13. The productivity shock immediately stimulates the aggregate output, TFP, and credit, as well as fundamental credit. On impact, the shock dampens the net flows to creditors in exchange for a sequence of positive net flows in the future. This corresponds to the increase in the

²²See, e.g., [Khan and Thomas \(2013\)](#) and [Ottonello and Winberry \(2020\)](#).

fundamental credit. Note that there is barely any fluctuation in the non-fundamental component.

Next, we study news shocks, i.e., anticipated changes in future A_t . We assume $A_t = 1$ at $t = 0$, before it rises to $A_t = 1 + \epsilon_A$ at $t = 1$. From $t = 1$ to $t = 99$ the rise in A_t decays at $\rho_A = 0.8$ per year before it fully perishes at $t = 100$. Albeit the productivity rise occurs at $t = 1$, it is expected at $t = 0$. The lines in Figure A.14 depict impulse responses. The effects of the news shock are analogous to the productivity shock with a delay. The increase in aggregate credit is likewise completely driven by the increase in the fundamental credit.

4.2 Interpreting the Reduced-form Empirical Evidence

This section uses the structural model as a laboratory to provide plausible structural interpretations of Facts 1 and 2. Specifically, we apply the strategies adopted in the empirical section to counterfactual data simulated using our structural model.

Following Boppart et al. (2018), we use impulse responses as numerical derivatives to simulate data. We use rollover, productivity, and pledgeability shocks separately to simulate three sets of data. Small measurement errors are added to simulated variables to avoid the issue of dynamic singularity. Each simulated data set consists of the same variables of the same sample size as in the empirical study.

The red lines in Figure A.15 plot the IRFs to a positive debt innovation, estimated by applying our empirical strategy to 1000 simulated data based on rollover shocks. The solid red lines are the median IRFs out of the 1000 simulations, and the dotted red lines indicate the 95% probability intervals. For comparison, the dashed blue lines in Figure A.15 re-plot the effects of a positive rollover shock shown in Figure 3. The IRFs plotted in Figure A.15 are similar to those obtained in the data.

Figure A.16 and A.17 respectively plot the simulation results based on productivity shocks and pledgeability shocks. The results are consistent with conventional wisdom: a credit boom is followed by significant increases in future surplus.

Our analysis demonstrates that credit bubble shocks can rationalize puzzling empirical observations.

5 Conclusion

This paper assesses the conventional view that a firm's credit equals its fundamental credit, i.e., the NPV of future net flows to creditors, by inspecting the co-movement between current liabilities and future net flows to creditors. Data reveals that a credit boom is *not* followed by increases in future net flows to creditors. More specifically, the net flows to creditors drop in the short run before converging back to their initial level. Moreover, interest rates barely change.

To interpret the puzzling empirical findings, we develop a model with financial frictions and heterogeneous firms, allowing for credit bubbles. We show that shocks to credit bubbles lead to procyclical aggregate credit and countercyclical fundamental credit. We demonstrate that the theoretical results are driven by firms' precautionary saving motives arising from financial frictions.

We focus on the macro consequences of credit bubbles rather than their origin. To this end, we model credit bubbles in an environment with rational and informed investors. However, it is worth noting that future research exploring the micro-foundations of credit bubbles could offer valuable insights in this area.

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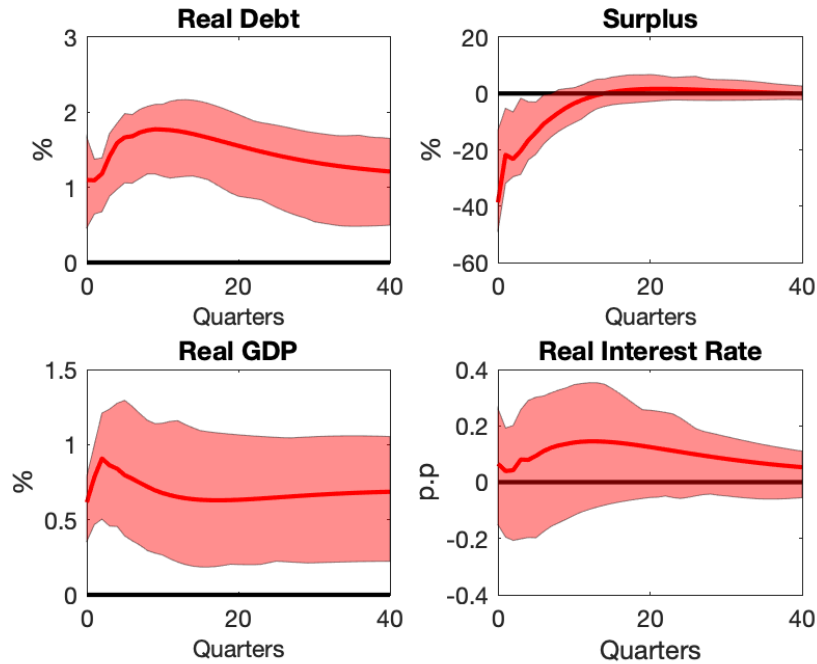
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Online Appendix to “Rolled-over Credit Cycles”

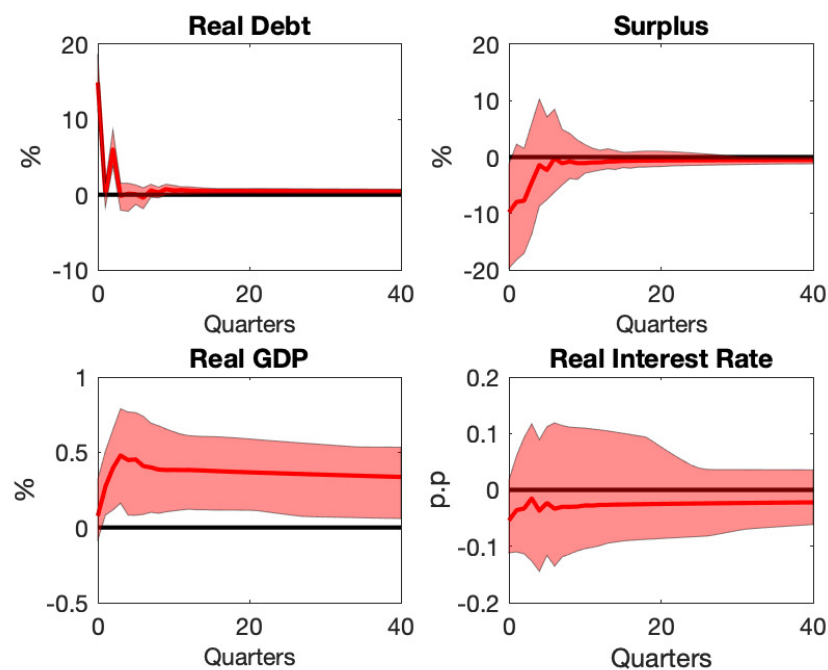
1 Figures

Figure A.1: Responses to Total Liabilities Innovation (Max-share): Pre-2007Q4 Sample



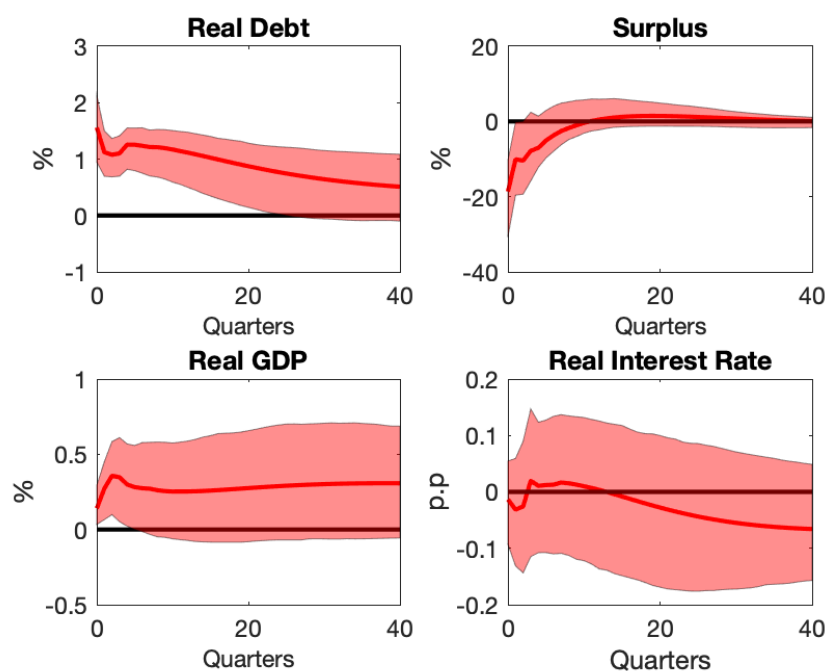
Note: This figure plots the impulse response functions to a positive max-share innovation to total liabilities using the pre-2007Q4 sample. The max-share innovation is constructed as the linear combination of reduced-form residuals that maximizes the fluctuations in total liabilities at the business cycle frequency. The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following [Kilian \(1998\)](#)'s method.

Figure A.2: Responses to Total Debt Innovation (Max-share)



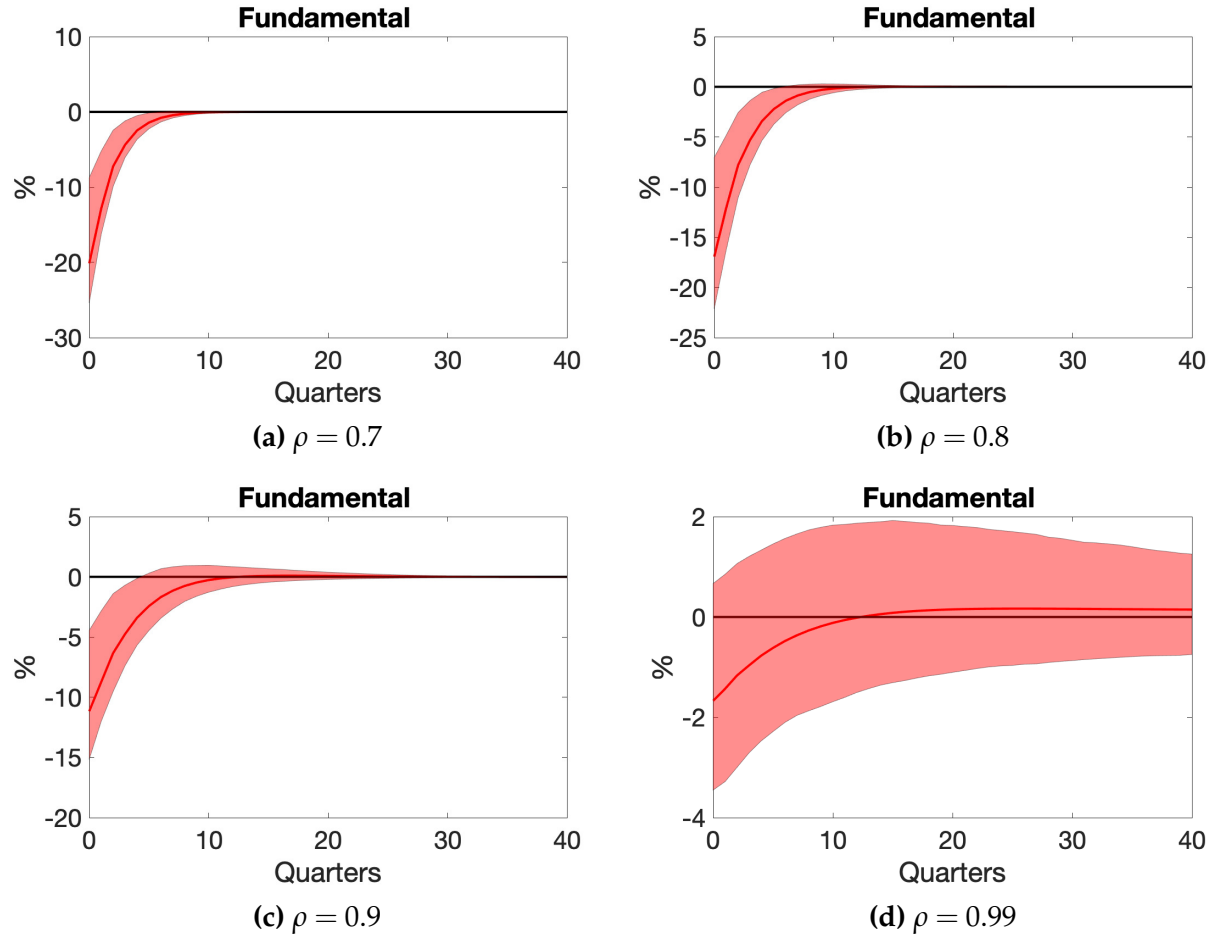
Note: This figure plots the impulse response functions to a positive max-share innovation to the total debt. The max-share innovation is constructed as the linear combination of reduced-form residuals that maximizes the fluctuations in the total debt at the business cycle frequency. The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following Kilian (1998)'s method.

Figure A.3: Responses to Total Liabilities Innovation (Zero Restrictions)



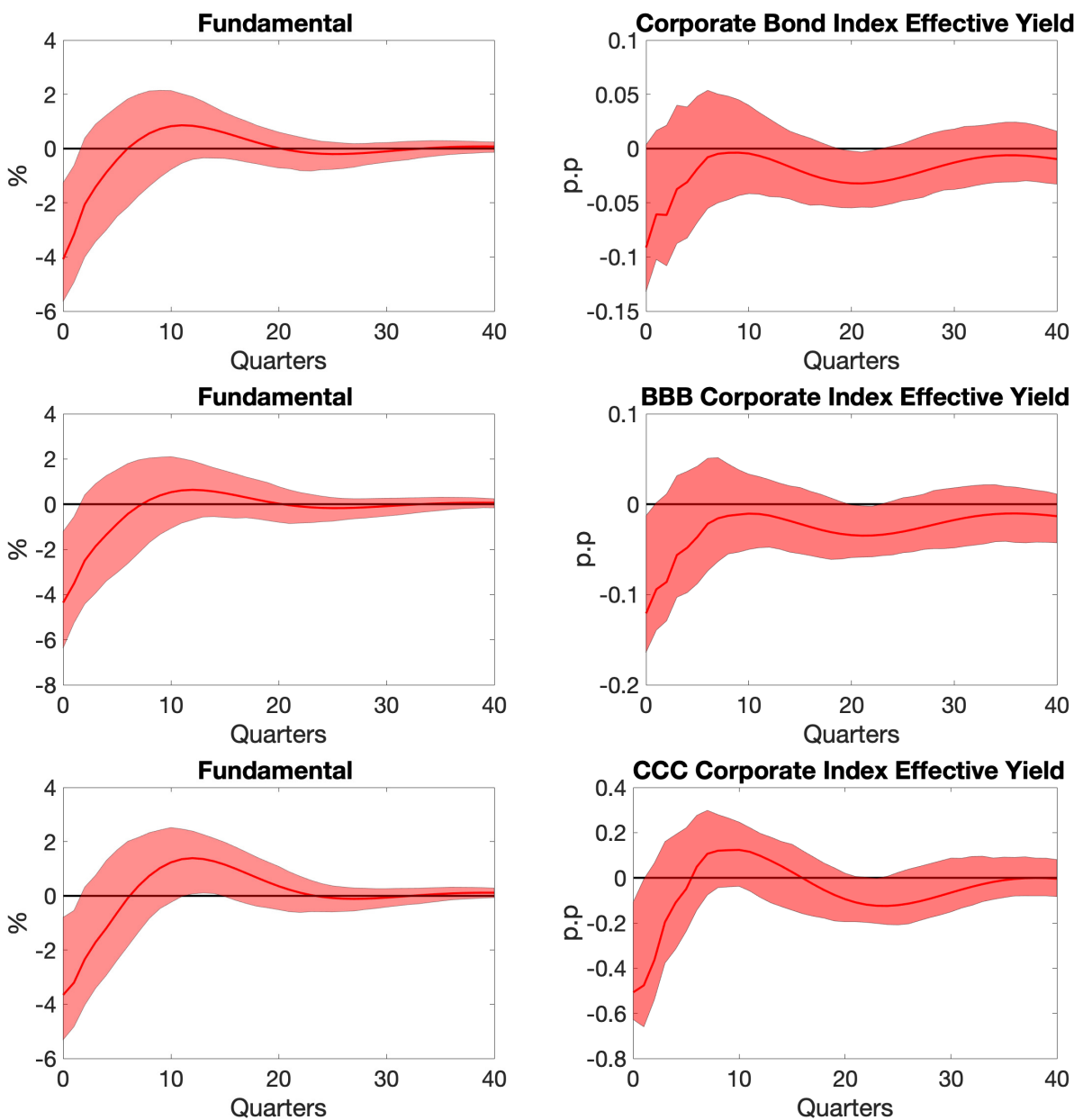
Note: This figure plots the impulse response functions to a positive innovation to total liabilities, constructed using the recursive identification strategy. The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following [Kilian \(1998\)](#)'s method.

Figure A.4: Responses of the NPV of Surpluses to total liabilities innovations: Alternative ρ



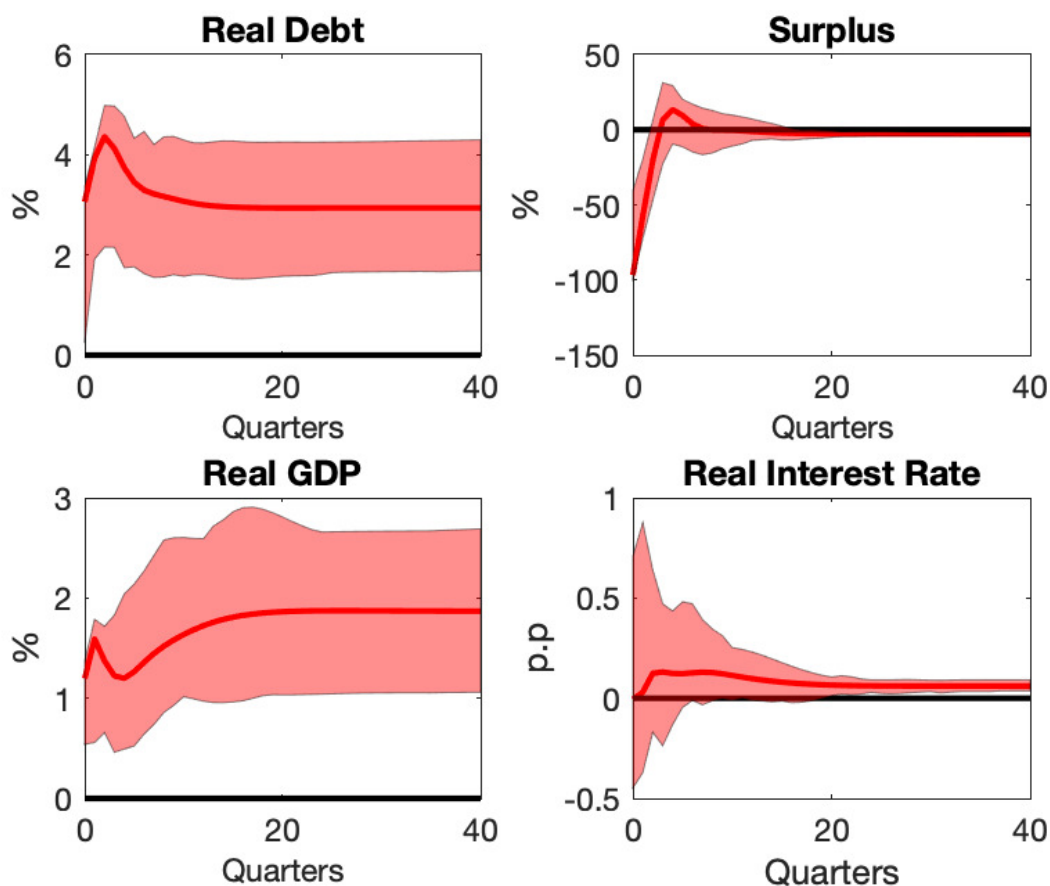
Note: This figure plots responses of the NPV of surpluses to total liabilities innovations under alternative calibrations of ρ . The IRFs are constructed using the IRFs of surpluses and the interest rate, according to equation (9). The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following [Kilian \(1998\)](#)'s method.

Figure A.5: Responses of the NPV of Surpluses to total liabilities innovations: Alternative r_t



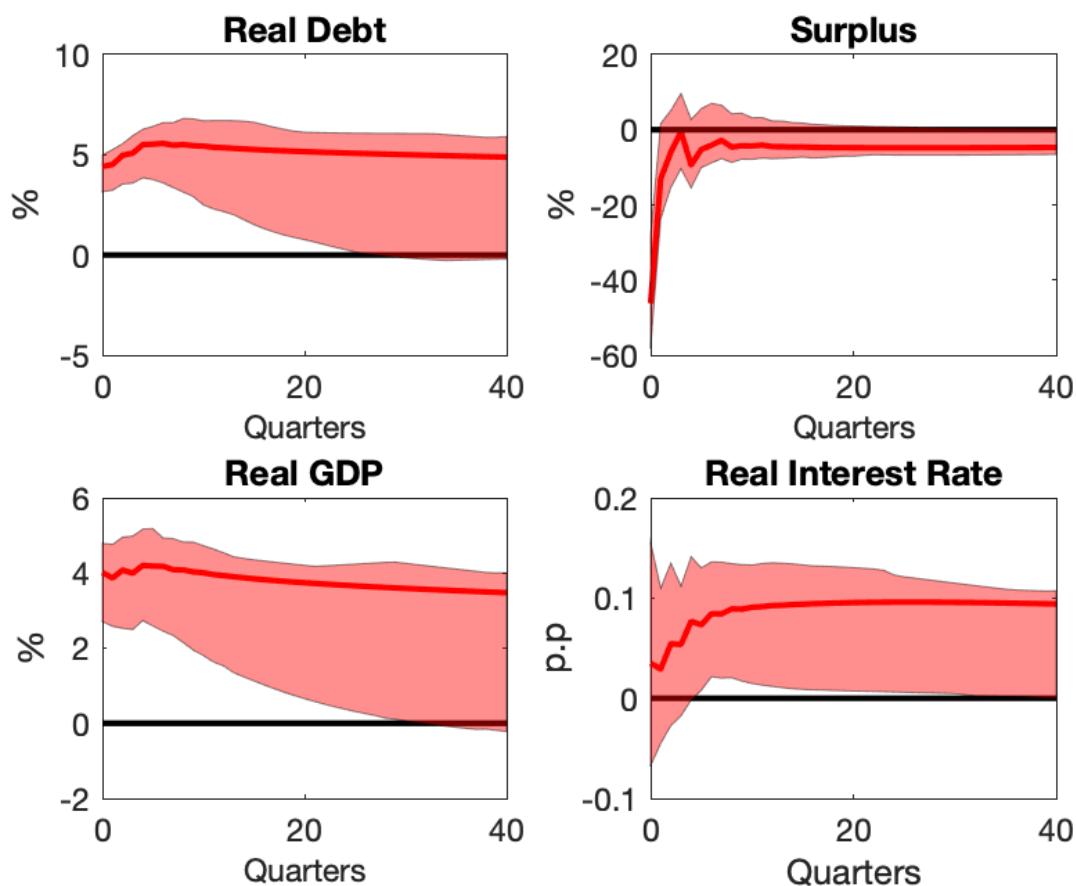
Note: This figure plots responses of the NPV of surpluses to total liabilities innovations constructed based on estimated VARs that include alternative measures of real interest rates. The IRFs are constructed using the IRFs of surpluses and the interest rate, according to equation (9). The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following Kilian (1998)'s method.

Figure A.6: Responses to Total Liabilities Innovation: Alternative Measure of Surpluses



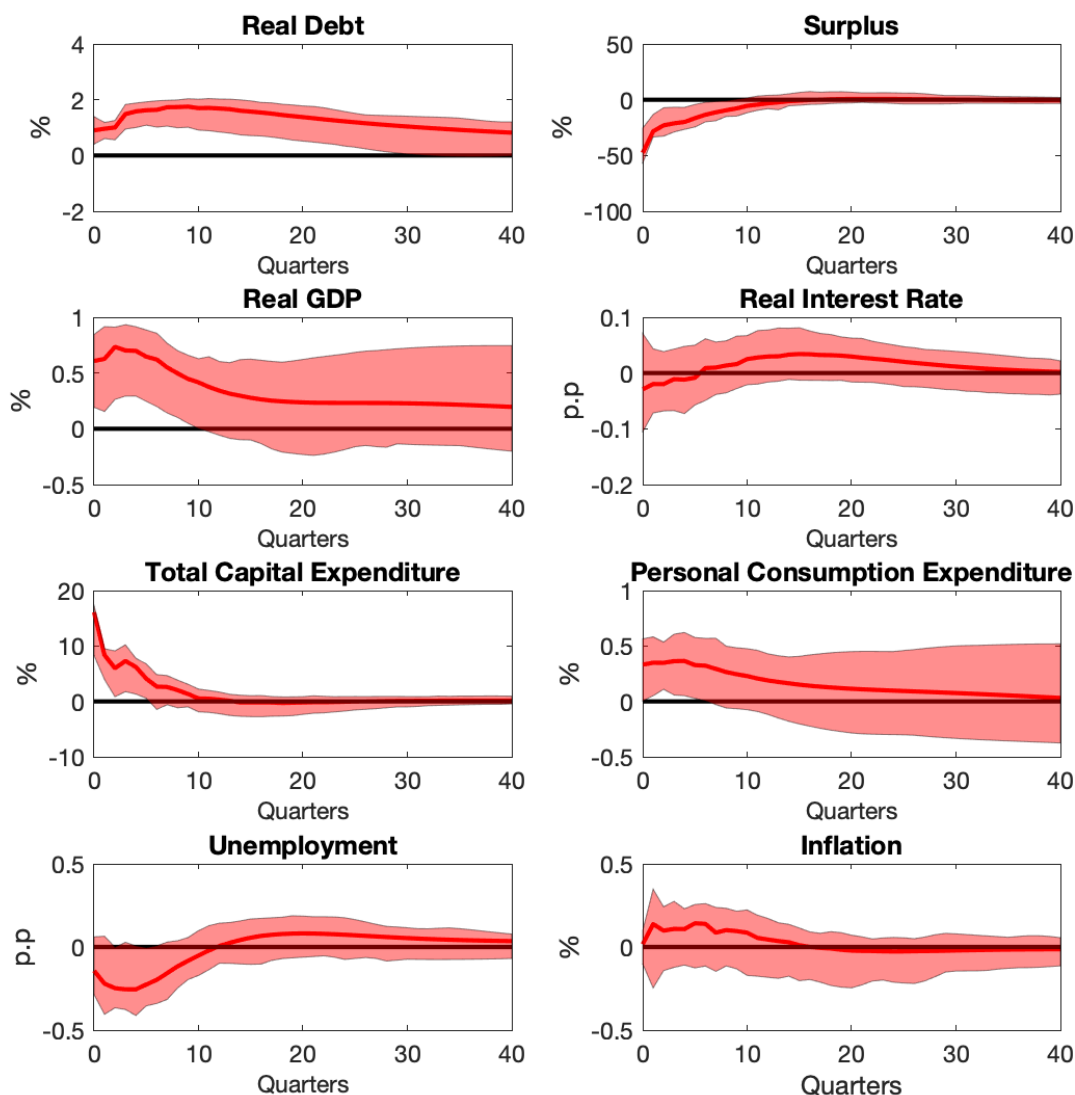
Note: This figure plots the impulse response functions to a positive max-share innovation to total liabilities using an alternative measure of surpluses. The max-share innovation is constructed as the linear combination of reduced-form residuals that maximizes the fluctuations in total liabilities at the business cycle frequency. The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following Kilian (1998)'s method.

Figure A.7: Responses to Total Liabilities Innovation (Max-share): Re-scaled by Equity



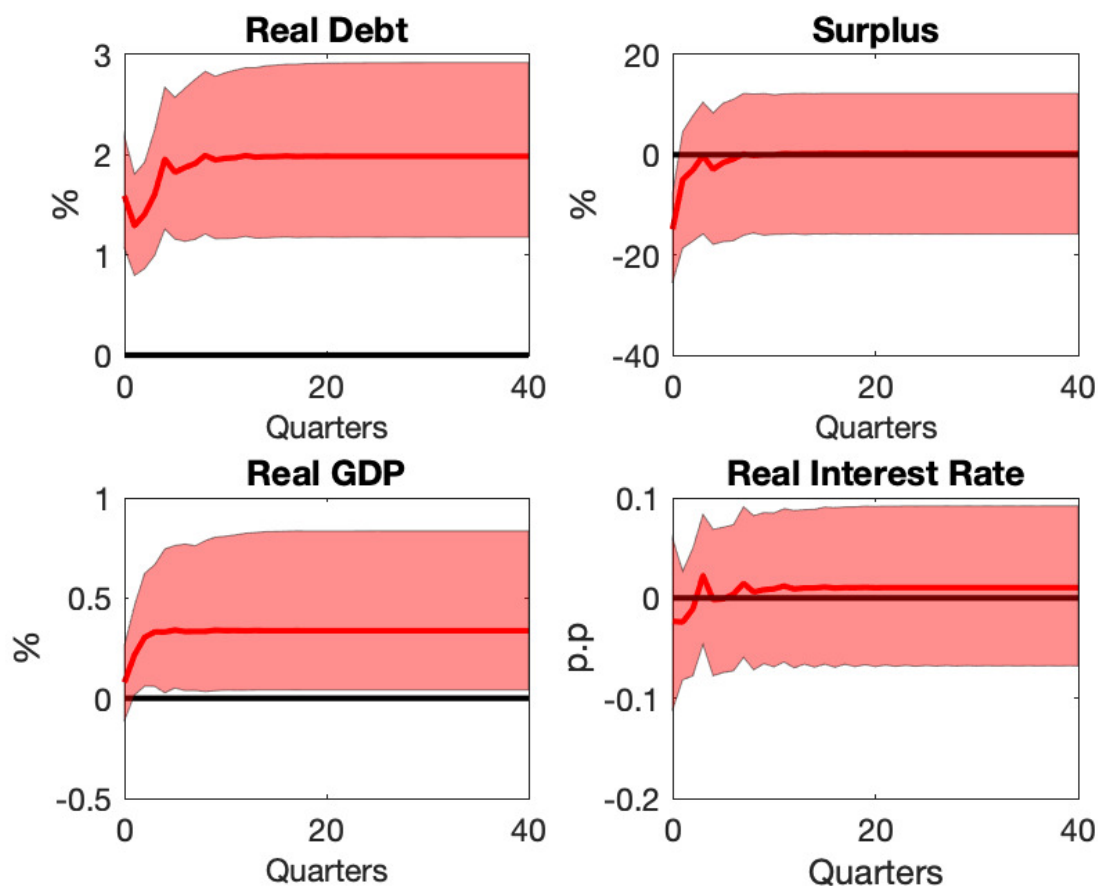
Note: This figure plots the impulse response functions to a positive max-share innovation to total liabilities based on a VAR model using variables re-scaled by the total equity. The max-share innovation is constructed as the linear combination of reduced-form residuals that maximizes the fluctuations in total liabilities at the business cycle frequency. The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following [Kilian \(1998\)](#)'s method.

Figure A.8: Responses to Total Liabilities Innovation (Max-share): Bigger VAR



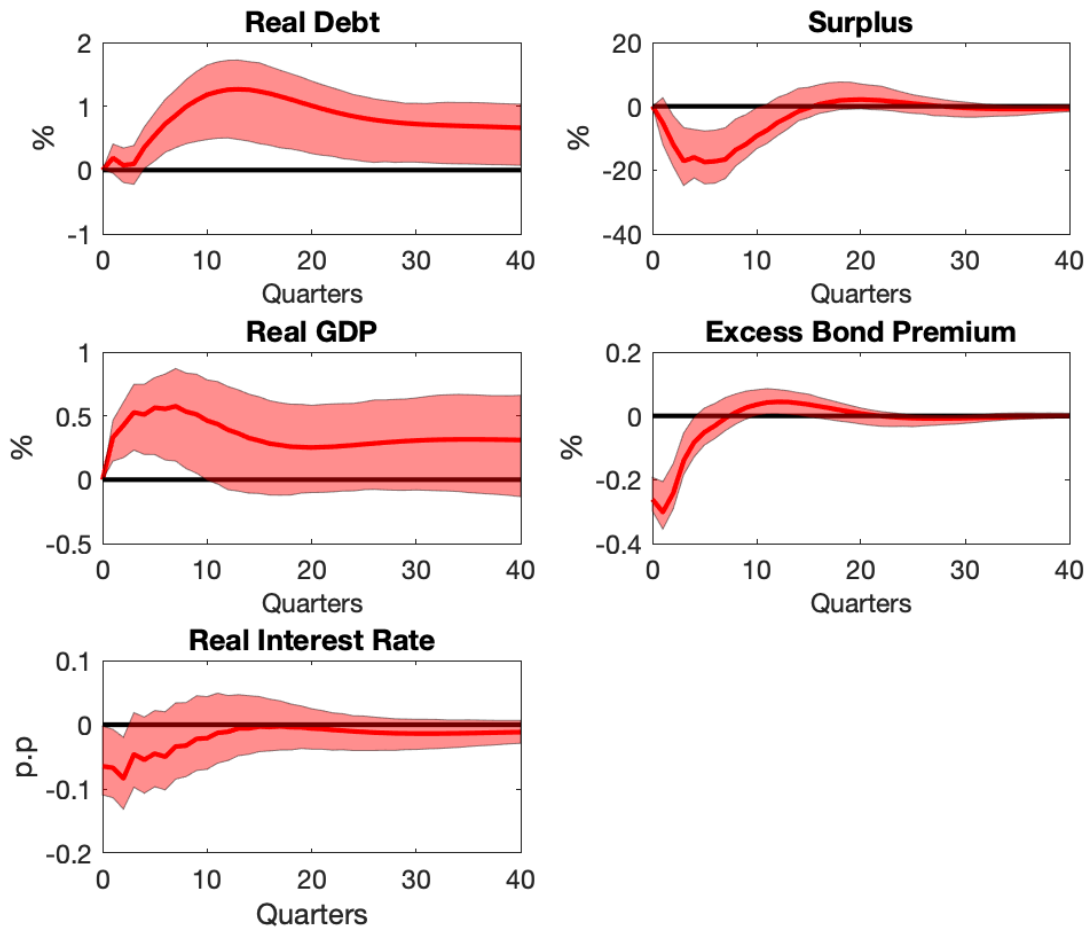
Note: This figure plots the impulse response functions to a positive max-share innovation to total liabilities based on a bigger VAR model that includes eight variables. The max-share innovation is constructed as the linear combination of reduced-form residuals that maximizes the fluctuations in total liabilities at the business cycle frequency. The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following Kilian (1998)'s method.

Figure A.9: Responses to Total Debt Innovation (Max-share): VAR in the First Difference



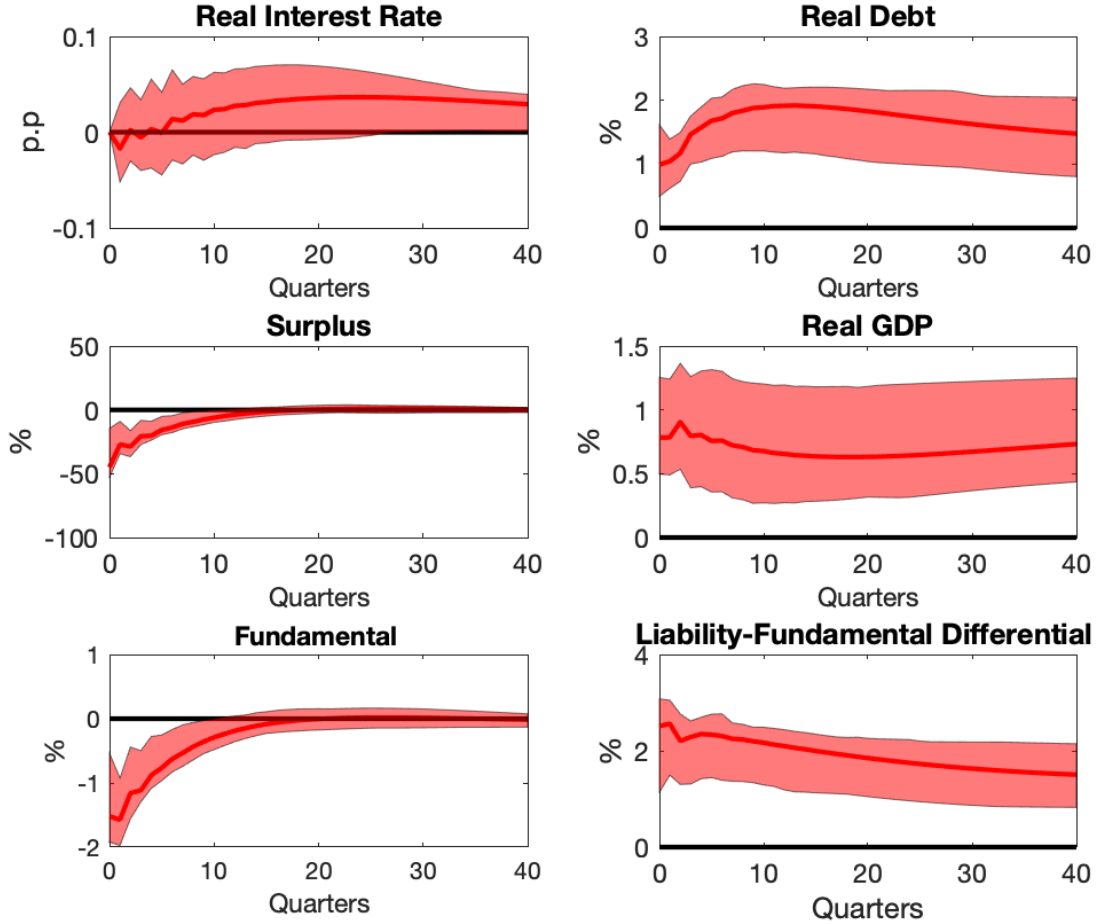
Note: This figure plots the impulse response functions to a positive max-share innovation to total liabilities based on a VAR model using variables in first difference. The max-share innovation is constructed as the linear combination of reduced-form residuals that maximizes the fluctuations in total liabilities at the business cycle frequency. The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following Kilian (1998)'s method.

Figure A.10: Responses to Excess Bond Premium Shocks



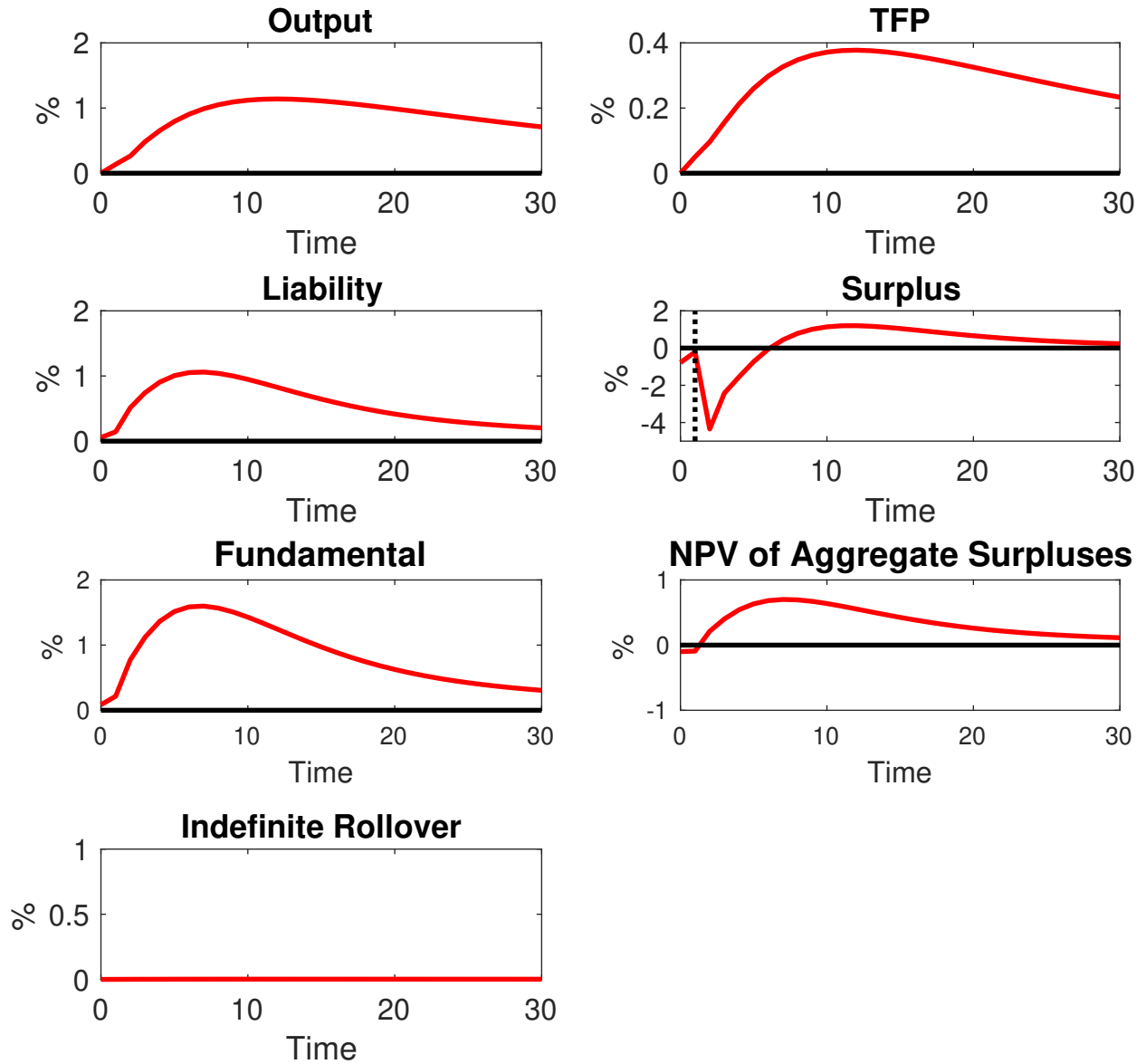
Note: This figure plots the impulse response functions to expansionary excess bond premium shocks. The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following [Kilian \(1998\)](#)'s method.

Figure A.11: Responses to Non-Fundamental Shocks in the Credit Market



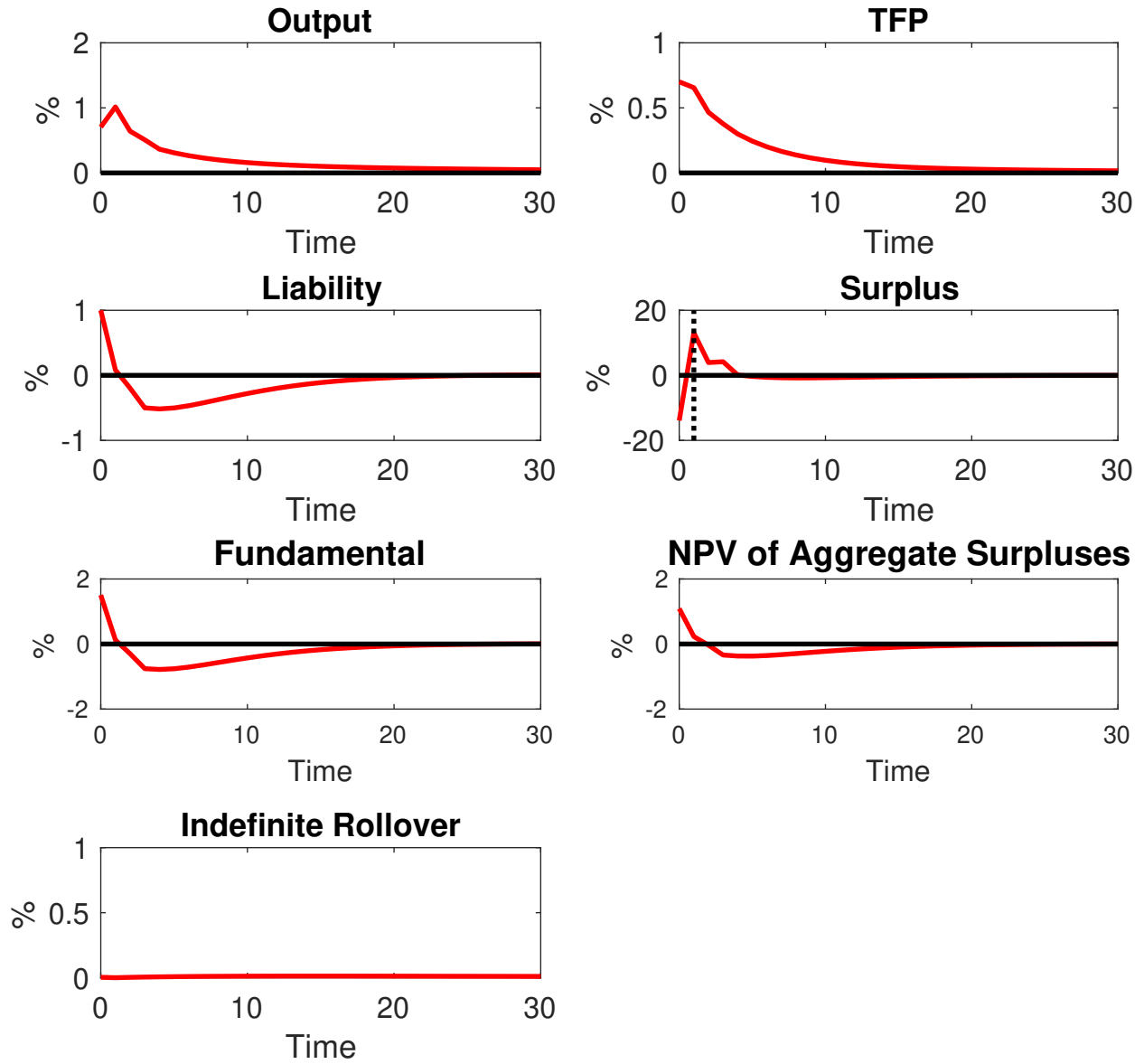
Note: This figure plots the impulse response functions to expansionary non-fundamental shocks in the credit market. Our identification strategy relies on the assumption that once controlled for shocks that affect the real interest rate, the shock that maximizes the forecast error variance decomposition of the liability-fundamental differential in the subsequent periods is a non-fundamental shock in the credit market. The shaded areas indicate the 95% confidence bands constructed using the bootstraps technique. The point estimates and confidence bands are corrected for the small sample bias following [Kilian \(1998\)](#)'s method.

Figure A.12: Model Prediction: Responses to Pledgeability Shocks



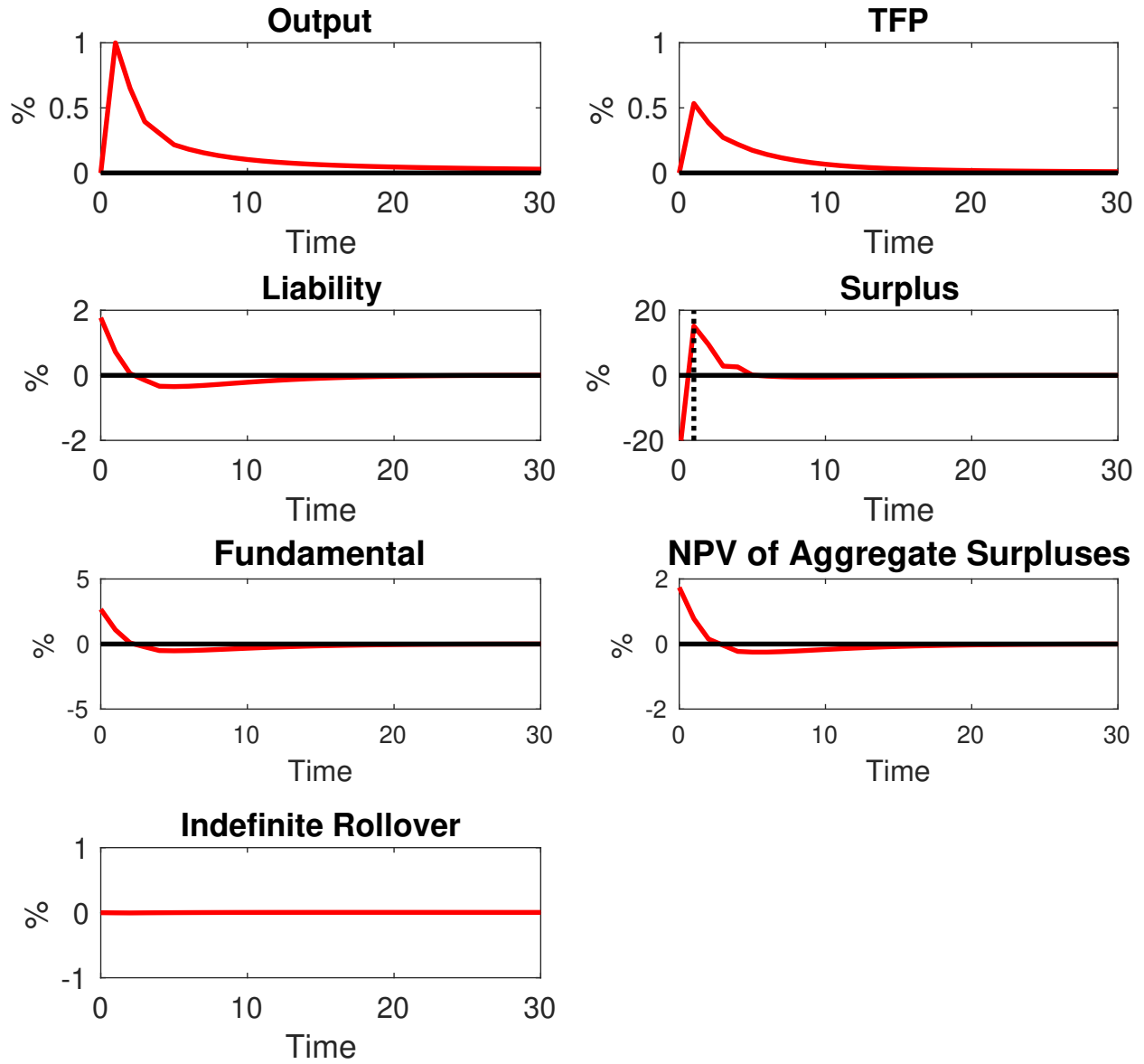
Note: This figure plots the IRFs to pledgeability shocks in the model.

Figure A.13: Model Prediction: Responses to Productivity Shocks



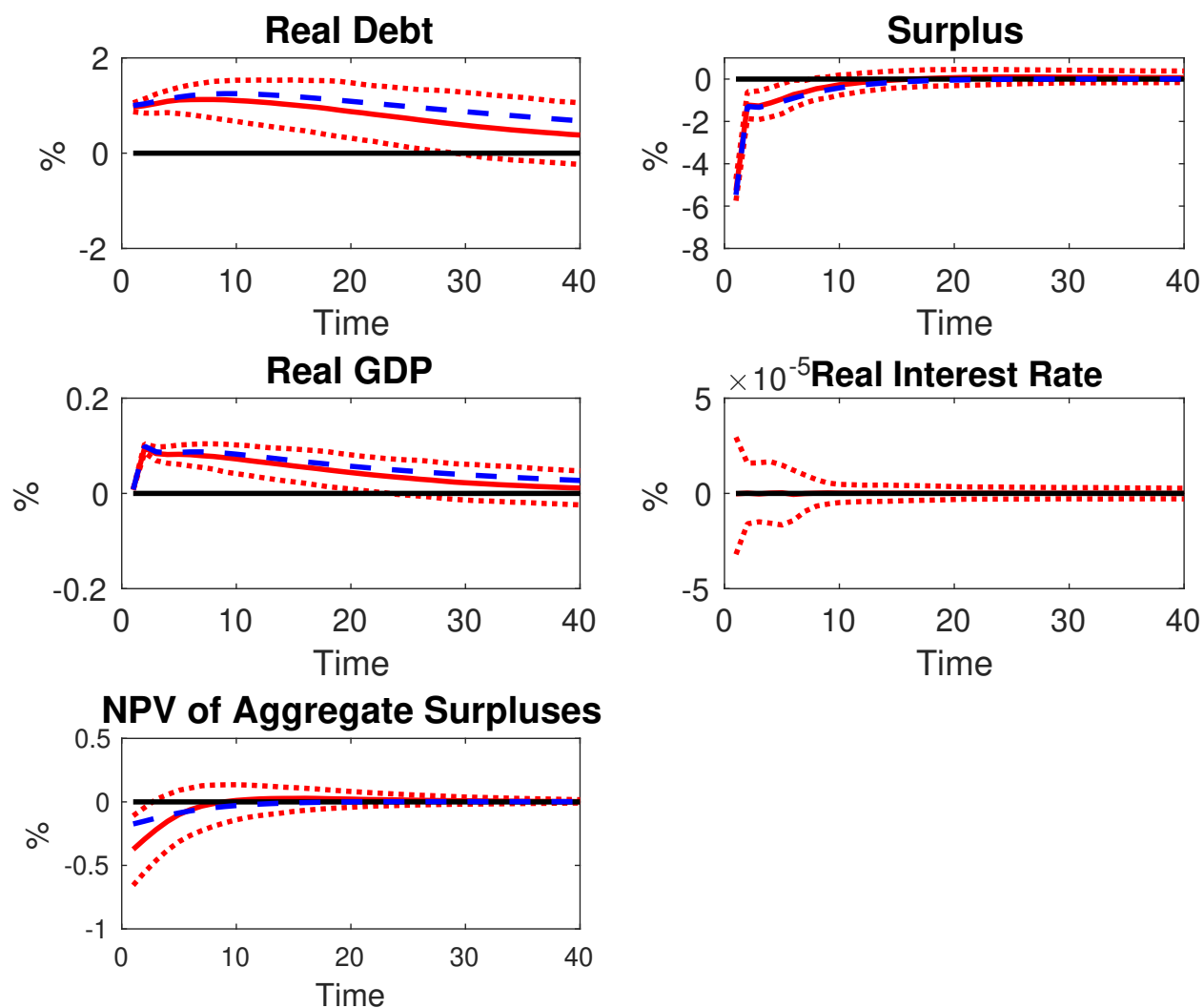
Note: This figure plots the IRFs to productivity shocks in the model.

Figure A.14: Model Prediction: Responses to News Shocks



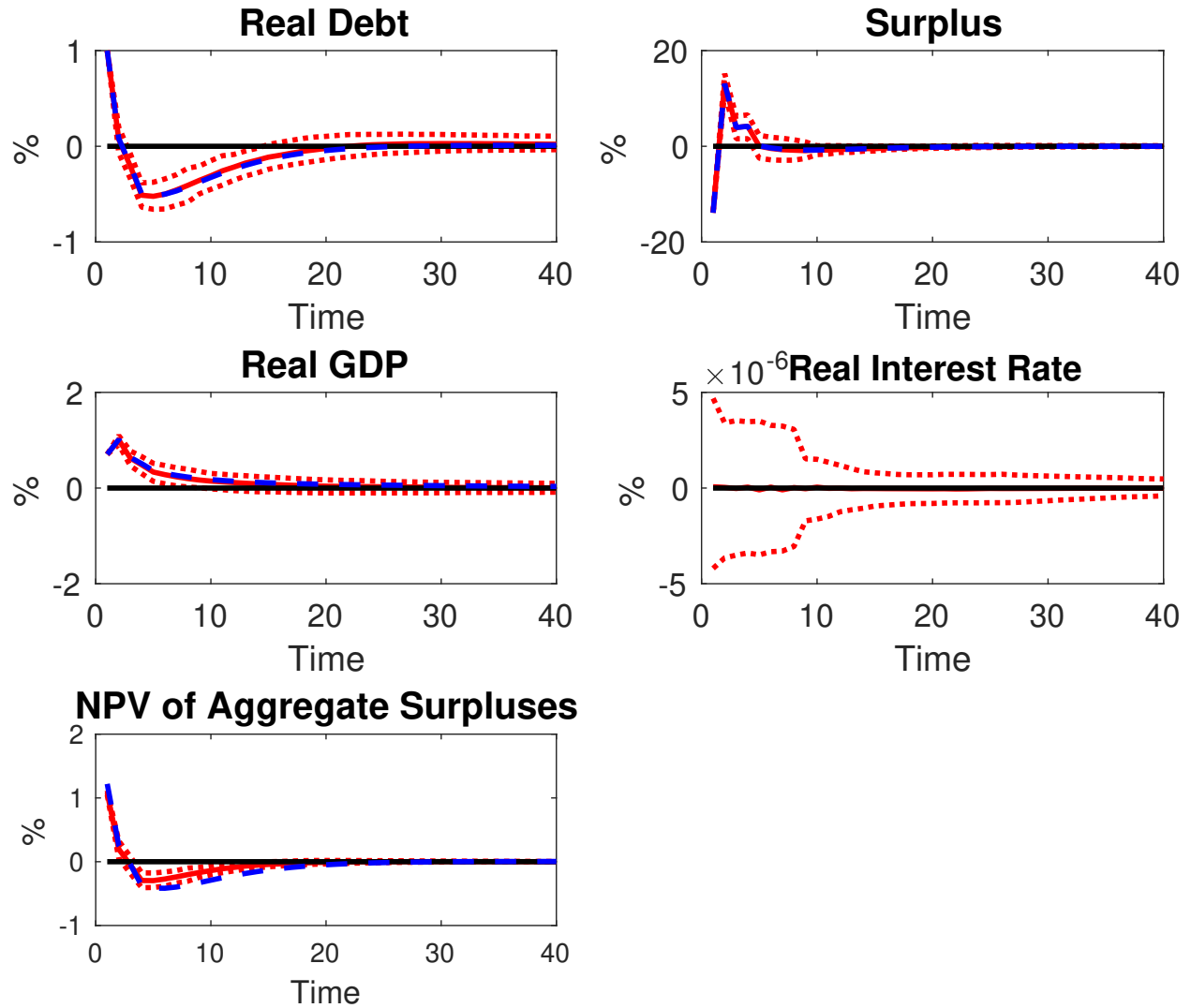
Note: This figure plots the IRFs to news shocks in the model.

Figure A.15: Model Simulation conditional on Rollover Shocks



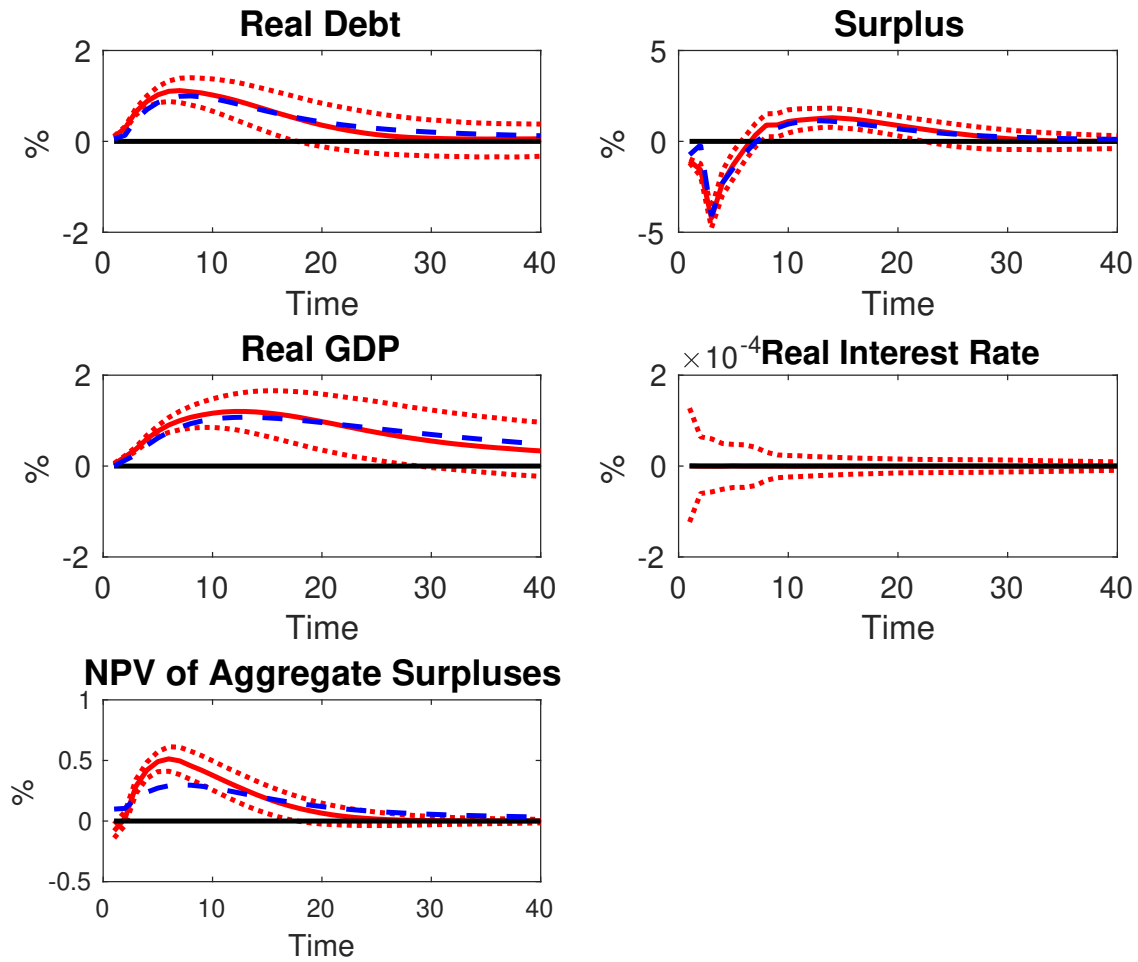
Note: The dashed blue lines plot the effects of a rollover shock in the model. The red lines plot the IRFs to a positive debt innovation, estimated by applying our empirical strategy to 1000 simulated data based on rollover shocks. The solid red lines are the median IRFs out of the 1000 simulations, and the dotted red lines indicate the 95% probability intervals.

Figure A.16: Model Simulation conditional on Productivity Shocks



Note: The dashed blue lines plot the effects of a productivity shock in the model. The red lines plot the IRFs to a positive debt innovation, estimated by applying our empirical strategy to 1000 simulated data based on productivity shocks. The solid red lines are the median IRFs out of the 1000 simulations, and the dotted red lines indicate the 95% probability intervals.

Figure A.17: Model Simulation conditional on Pledgeability Shocks



Note: The dashed blue lines plot the effects of a pledgeability shock in the model. The red lines plot the IRFs to a positive debt innovation, estimated by applying our empirical strategy to 1000 simulated data based on pledgeability shocks. The solid red lines are the median IRFs out of the 1000 simulations, and the dotted red lines indicate the 95% probability intervals.

2 Appendix for Empirical Analysis

2.1 Data

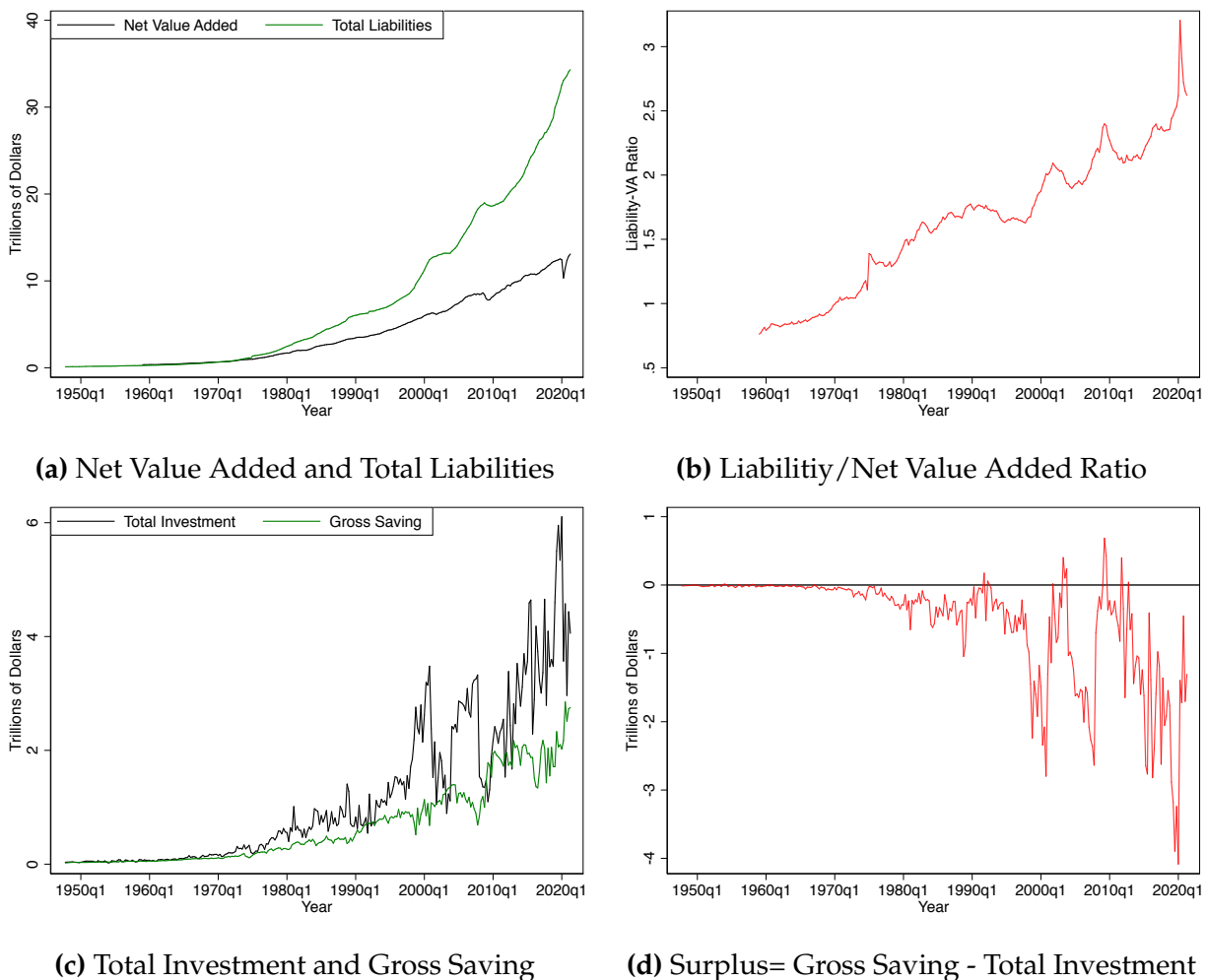


Figure A.18: Non-financial Firms in the United States

We use data from the Financial Accounts of the United States, previously known as the Flow of Funds Accounts. We combine flows of funds data for non-financial corporate businesses (Table F.103) and non-financial non-corporate businesses (Table F.104).

The total liabilities are the sum of the total liabilities (excluding equities) for non-financial corporate and non-financial non-corporate firms.

The surplus (S_t) is constructed as the "gross saving less net capital transfers paid" for both corporate business (Table F.103 line 8) and non-corporate non-financial business

(Table F.104 line 2) + "changes in equities" (Table F.103 line 43 and Table F.104 line 32) – "total capital expenditures" (Table F.103 line 8 plus Table F.104 line 4) – "new acquisition of financial assets" (Table F.103 line 15 plus Table F.104 line 8).

The total debt is the sum of the total debt for non-financial corporate and non-financial non-corporate firms. The total interest payment is the sum of interest paid for non-financial corporate and non-financial non-corporate firms. Figure A.18 plots the data we used from the Financial Accounts of the United States.

We use Gilchrist and Zakrajšek (2012)'s excess bond premium, which is updated by Favara et al. (2016).

The remaining variables are taken from FRED: the real gross domestic product (real GDP), GDP deflator, total capital expenditure, personal consumption expenditure, unemployment rate, Moody's Baa Corporate Bond Yield, the US corporate index effective yield, the BBB US corporate index effective yield, and the CCC & lower US high yield index effective yield. All nominal variables are re-scaled by the GDP deflator to obtain their real values.

2.2 Cash Flow Equation

This section builds upon Cochrane (2021) and Hamilton and Flavin (1986) to study firms' liabilities. We collect all non-financial firms' zero-coupon debt of a given maturity j into group j . Let $d_{j,t}$ denote the market value of a debt of the group j at the end of period t , and $q_{j,t}$ is the price of the debt j at time t . Let D_t denote the real market value of total non-financial firms' debt held by the market at the end of period t : $D_t \equiv \sum_j q_{j,t} d_{j,t}$.

Investors' return (R_{t+1}) of holding the portfolio of non-financial firms' debt depends on prices (taking into account potential default). The return of such a portfolio is defined as:

$$R_{t+1} \equiv \frac{\sum_j q_{j,t+1} d'_{j,t}}{\sum_j q_{j,t} d_{j,t}}, \quad (42)$$

where $d'_{j,t}$ is the amount of outstanding debt at time t that are not defaulted.

The derivation starts with the following cash flow identity:

$$\sum_j q_{j,t+1} d'_{j,t} = S_{t+1} + \sum_j q_{j,t+1} d_{j,t+1}, \quad (43)$$

where S_{t+1} denotes the surplus of non-financial firms: gross saving plus changes in total equity (market value) minus total investment. The total investment includes both the capital investment and the financial investment.

Equation (42) implies $\sum_j q_{j,t+1} d'_{j,t} = R_{t+1} \sum_j q_{j,t} d_{j,t}$, plus this into the cash flow identity:

$$R_{t+1} \sum_j q_{j,t} d_{j,t} = S_{t+1} + \sum_j q_{j,t+1} d_{j,t+1}. \quad (44)$$

Using the notation of the real market value of total non-financial firms' debt leads to the equation (1) in the main text:

$$R_{t+1} D_t = S_{t+1} + D_{t+1}.$$

Note that allowing for non-zero coupon debt does not change the final equation, see [Hamilton and Flavin \(1986\)](#) for an alternative derivation that allows for coupon payments.¹

2.2.1 Log Linearization

To construct the effects of debt innovations on its fundamental component, we first log linearize the equation (1), then substitute it forward to obtain the log linearized fundamental as the discounted sum of future surpluses and interest rates.

Divide (1) by its counter-part at the BGP and take log on both hand side of the equation:

$$\hat{r}_{t+1} + \hat{d}_t = \log\left(\frac{S_{t+1}}{S + D} + \frac{D_{t+1}}{S + D}\right), \quad (45)$$

where S and D are the surplus and debt on the BGP and a variable with hat denotes the log deviation of the variable from its BGP. Because surplus can be negative in the data, we linearize the equation (1) in surplus in percentage deviation from its sample mean. To do so, first rewrite the previous equation as

$$\hat{r}_{t+1} + \hat{d}_t = \log\left(\tilde{s}_{t+1} \frac{S}{S + D} + \frac{D_{t+1}}{D} \frac{D}{S + D} + \frac{S}{S + D}\right), \quad (46)$$

¹Interest payment does not enter directly into the right hand side of the equation (1), see [Hamilton and Flavin \(1986\)](#) for detailed explanations in the context of government bond.

where $\tilde{s}_{t+1} \equiv \frac{S_{t+1}-S}{S}$ is the surplus in percentage deviation from its sample mean. Replace the right-hand side with its first-order Taylor approximation and neglect the constant terms

$$\hat{r}_{t+1} + \hat{d}_t = \frac{S}{S+D} \tilde{s}_{t+1} + \frac{D}{S+D} \hat{d}_{t+1}, \quad (47)$$

To be able to implement the equation to the data, we expand the equation around the point where surplus and debt grow at the common rate. The previous equation can be re-written as:

$$\hat{r}_{t+1} + \hat{d}_t = (1 - \rho) \tilde{s}_{t+1} + \rho \hat{d}_{t+1}, \quad (48)$$

where $\rho \equiv \frac{1}{sd+1}$ and sd denotes the surplus-to-debt ratio on the BGP. Substitute equation (48) forward, and impose the no-Ponzi-game condition to obtain the fundamental component of the debt:

$$\hat{f}_t = \sum_{h=0}^{\infty} \rho^h [(1 - \rho) \tilde{s}_{t+h+1} - \hat{r}_{t+h+1}], \quad (49)$$

2.3 Aggregation

This section derives the *aggregate* cash flow Equation (7) and the aggregate fundamental component Equation (9) with heterogeneous firms.

Following the derivations illustrated in Appendix 2.2, one can show that the cash flow equation holds at the firm level:

$$R_{jt} D_{jt-1} - D_{jt} = S_{jt}, \quad (50)$$

where D_{jt} , R_{jt} , and S_{jt} respectively denote the outstanding debt, the average interest rate on the debt, and the surplus for firm j . Note that Equation (50) also holds for new firms, whose $D_{jt-1} = 0$. Equation (50) holds in expectation:

$$D_{jt} = E_t \left[\frac{1}{R_{jt+1}} (S_{jt+1} + D_{jt+1}) \right]. \quad (51)$$

The *aggregate* outstanding debt (liability) is equal to

$$\begin{aligned} D_t &= \Sigma_j D_{jt} = E_t \left\{ \Sigma_j \left[\frac{1}{R_{jt+1}} (S_{jt+1} + D_{jt+1}) \right] \right\} \\ &= E_t [(S'_{t+1} + D'_{t+1})], \end{aligned} \quad (52)$$

where $S'_{t+1} \equiv \Sigma_j \frac{1}{R_{jt+1}} S_{jt+1}$, $D'_{t+1} \equiv \Sigma_j \frac{1}{R_{jt+1}} D_{jt+1}$.

Next, log-linearize Equation (52) to obtain

$$\hat{d}_t = (1 - \rho) \hat{s}'_{t+1} + \rho \hat{d}'_{t+1}, \quad (53)$$

where $\rho \equiv \frac{D'}{S' + D'}$.

Note that

$$\begin{aligned} \log S'_{t+1} &= \log S_{t+1} + \log \Sigma_j \frac{1}{R_{jt+1}} \frac{S_{jt+1}}{S_{t+1}} = \log S_{t+1} + \log \Sigma_j \frac{1}{R_{jt+1}} \frac{S_{jt+1}}{S_{t+1}} \\ &= \log S_{t+1} - \log R_t^s \end{aligned} \quad (54)$$

where $R_{t+1}^s \equiv \left(\Sigma_j \frac{1}{R_{jt+1}} \frac{S_{jt+1}}{S_{t+1}} \right)^{-1}$. Hence

$$\hat{s}'_{t+1} = \hat{s}_{t+1} - \hat{r}_{t+1}^s. \quad (55)$$

Similarly, define $R_{t+1}^d = \left(\Sigma_j \frac{1}{R_{jt+1}} \frac{D_{jt+1}}{D_{t+1}} \right)^{-1}$, and we have

$$\hat{d}'_{t+1} = \hat{d}_{t+1} - \hat{r}_{t+1}^d. \quad (56)$$

We can thus rewrite (53) into

$$\hat{d}_t = (1 - \rho) \tilde{s}_{t+1} + \rho \hat{d}_{t+1} - \hat{r}_{t+1}. \quad (57)$$

where $\hat{r}_{t+1} \equiv [(1 - \rho) \hat{r}_{t+1}^s + \rho \hat{r}_{t+1}^d]$.

The equation (57) is similar to the log linearized version of equation (1). In a complex economy with heterogeneous interest rates across firms, the aggregate cash flow equation features an aggregate interest rate, which is a weighted average of firm-level interest rates.

Solve equation (57) forward, and impose the no-Ponzi-game condition to obtain the

aggregate fundamental component:²

$$\hat{f}_t = \sum_{h=0}^{\infty} (\rho)^h [(1 - \rho)\tilde{s}_{t+h+1} - \hat{r}_{t+h+1}], \quad (58)$$

Discussions With heterogeneous firms, r_{t+1} reflects the average interest rate where the weights of different firms' interest rates depend on the relative sizes of firms' surplus and debt. However, the functional form that links the aggregate debt with current and future aggregate surplus remains unchanged. Note that we do not directly observe r_{t+1} and ρ in the data. In the robustness checks, we verify that our empirical results are robust to alternative measures of ρ and R_t .

Note that the derivation allows for the entry and default of firms.

3 Model Appendix

3.1 Proof of Proposition 1

We prove this proposition for incumbent firms. The same reasoning can be applied to the optimization problem of potential entrants. Suppose that the optimal path for firm j can be represented by policy functions $k_t^*(\varphi_{jt}, \lambda_{jt})$, $F_t^*(\varphi_{jt}, \lambda_{jt})$, $B_t^*(\varphi_{jt}, \lambda_{jt})$ for $\forall t$, which respectively denote policy functions for k_{jt+1} , F_{jt+1} , B_{jt+1} .³ Moreover, at τ , $q_\tau^B(\varphi_{j\tau}, K_{j\tau+1}, F_{j\tau+1}, B_{j\tau+1}) B_{j\tau+1} < B_{j\tau}$. As in the main text, the corresponding value of firm j is denoted by $V_t(\varphi_{jt}, \lambda_{jt})$. We label this path as Path 1. Note that idiosyncratic states can be sufficiently summarized by φ_{jt} , x_{jt} and B_{jt} . Therefore hereafter we replace λ_{jt} with x_{jt} and B_{jt} .

We need to prove that, when choosing $q_\tau^B(\varphi_{j\tau}, K_{j\tau+1}, F_{j\tau+1}, B_{j\tau+1}) B_{j\tau+1} = B_{j\tau}$, the net present value of expected dividends is at least as great as $V_\tau(\varphi_{j\tau}, x_{j\tau}, B_{j\tau})$. Consider an alternative path: at τ , the firm chooses $q_\tau^B(\varphi_{j\tau}, K_{j\tau+1}, F_{j\tau+1}, B_{j\tau+1}) B_{j\tau+1} = B_{j\tau}$, $F_{j\tau+1} = F_\tau^*(\varphi_{j\tau}, x_{j\tau}, B_{j\tau}) + B_\tau^*(\varphi_{j\tau}, x_{j\tau}, B_{j\tau}) - B_{j\tau+1}$, $k_{j\tau+1} = k_\tau^*(\varphi_{j\tau}, x_{j\tau}, B_{j\tau})$; at $\tau + 1$, the firm chooses $B_{j\tau+2} = B_{\tau+1}^*(\varphi_{j\tau+1}, x_{j\tau+1}, B_\tau^*(\varphi_{j\tau}, x_{j\tau}, B_{j\tau}))$, $F_{j\tau+2} = F_{\tau+1}^*(\varphi_{j\tau+1}, x_{j\tau+1}, B_\tau^*(\varphi_{j\tau}, x_{j\tau}, B_{j\tau}))$,

²no-Ponzi-game condition implies that $F_{jt} = D_{jt}$ for $\forall t, j$. Hence $F_t = D_t$, i.e., $f_t = d_t$.

³Throughout this paper, we assume that there exists unique solution to policy functions and value functions. We follow the typical procedure in the literature by verifying that our iterative numerical solution leads to the same results, given different initial guesses. We also assume that the probability of repayment is greater than zero over the state space, which can also be verified in the numerical exercise.

$k_{j\tau+2} = k_{\tau+1}^* (\varphi_{j\tau+1}, x_{j\tau+1}, B_{\tau}^* (\varphi_{j\tau}, x_{j\tau}, B_{j\tau}))$; $\forall t > \tau + 1$, $k_{jt+1} = k_t^* (\varphi_{jt}, x_{jt}, B_{jt})$, $F_{jt+1} = F_t^* (\varphi_{jt}, x_{jt}, B_{jt})$, $B_{jt+1} = B_t^* (\varphi_{jt}, x_{jt}, B_{jt})$. We label this path as Path 2.

It is straightforward to prove that, compared with the supposedly optimal Path 1, Path 2 generates exactly the same sequence of dividends (and therefore the same default probability) for all $t \geq \tau$, regardless of the realization of φ_{jt} . In fact, $\forall t \geq \tau + 1$, control variables k_{jt+1} , F_{jt+1} , B_{jt+1} are the same across two paths for any realization of φ_{jt} , μ_t .

Moreover, we can verify that along Path 2, at $\tau + 1$, $q_{j\tau+1}^B B_{j\tau+2} \leq B_{j\tau+1}$. Along Path 2,

$$B_{j\tau+1} = \frac{B_{j\tau}}{q_{\tau}^B (\varphi_{j\tau}, K_{j\tau+1}, F_{j\tau+1}, B_{j\tau+1})}.$$

According to Equation (22), the price schedule $q_{\tau}^B (\varphi_{j\tau}, K_{j\tau+1}, F_{j\tau+1}, B_{j\tau+1})$ is pinned down by the default probability at $\tau + 1$. Since the default probabilities are the same at $\tau + 1$ across the two paths, the price schedules are the same at $\tau + 1$ across the two paths. Therefore along Path 2 we have

$$B_{j\tau+1} = \frac{B_{j\tau}}{q_{\tau}^B (\varphi_{j\tau}, K_{j\tau+1}, F_{j\tau+1}, B_{j\tau+1})} > B_{\tau}^* (\varphi_{j\tau}, x_{j\tau}, B_{j\tau}).$$

Along Path 1, at $\tau + 1$, $q_{j\tau+1}^B B_{j\tau+2} \leq B_{j\tau+1} = B_{\tau}^* (\varphi_{j\tau}, x_{j\tau}, B_{j\tau})$. Furthermore, $q_{j\tau+1}^B B_{j\tau+2}$ are the same across Path 1 and Path 2. We can thus verify that along Path 2, $q_{j\tau+1}^B B_{j\tau+2} \leq B_{\tau}^* (\varphi_{j\tau}, x_{j\tau}, B_{j\tau}) < B_{j\tau+1}$.

To sum up, when choosing $q_{\tau}^B (\varphi_{j\tau}, K_{j\tau+1}, F_{j\tau+1}, B_{j\tau+1}) B_{j\tau+1} = B_{j\tau}$, the net present value of expected dividends is at least as great as $V_{\tau} (\varphi_{j\tau}, x_{j\tau}, B_{j\tau})$. Note that we construct Path 2 so that it is simply as good as the supposedly optimal path, yet Path 2 is not necessarily the actual optimal path.

An incumbent firm faces the constraint $q_{\tau}^B (\varphi_{j\tau}, K_{j\tau+1}, F_{j\tau+1}, B_{j\tau+1}) B_{j\tau+1} \leq B_{j\tau}$. The proof above applies to the optimization problem of potential entrants if we replace the right-hand-side upper bound $B_{j\tau}$ with $s_{j\tau} B_0$. In fact, the reasoning also works if we replace the upper bound $B_{j\tau}$ with an arbitrary number $\bar{B}$: at τ , if firm j faces the constraint

$$q_{\tau}^B (\varphi_{j\tau}, K_{j\tau+1}, F_{j\tau+1}, B_{j\tau+1}) B_{j\tau+1} \leq \bar{B},$$

and B_{jt+1} follows (23) $\forall t > \tau$ and can be rolled over indefinitely, it is optimal to choose

$$q_{\tau}^B (\varphi_{j\tau}, K_{j\tau+1}, F_{j\tau+1}, B_{j\tau+1}) B_{j\tau+1} = \bar{B}.$$

In a nutshell, it is always optimal to maximize the borrowing in indefinite rollover debts B .

3.2 Define an Equilibrium

A recursive competitive bubbly equilibrium consists of: i) value functions $V_t(\varphi, \tilde{x})$ and $V_t^e(\varphi)$ ii) policy functions $k_t^*(\varphi, \tilde{x})$, $F_t^*(\varphi, \tilde{x})$ and $B_t^*(\varphi, \tilde{x}, B)$, iii) the measure of entrants $\zeta_t(\varphi, s)$, the measure of surviving firms $\eta_t(\varphi, \tilde{x}, B)$, iv) the initial bubble component B_0 , v) debt price schedule $q_t^F(\varphi, F', k')$, and vi) the investment return R_t , such that for all t ,

1. $V_t(\varphi_t, \tilde{x}_t)$ solves the optimization problem of incumbent firms, which is described by

$$V_t(\varphi, \tilde{x}) = \max_{F', k'} \left\{ \tilde{x} - k' + q_t^F(\varphi, F', k') F' + \frac{1}{R_{t+1}} \int V_{t+1}(\varphi', \tilde{x}') dJ(\varphi' | \varphi) \right\}, \quad (59)$$

$$\tilde{x} - k' + q_t^F(\varphi, F', k') F' \geq 0, \quad (60)$$

where $k_t^*(\varphi, \tilde{x})$ and $F_t^*(\varphi, \tilde{x})$ solve the optimal k' and F' respectively. $B_t^*(\varphi, \tilde{x}, B)$ follows

$$B_t^*(\varphi, \tilde{x}, B) = \frac{R_{t+1}}{p_{t+1}^n(\varphi, F', k')} B, \quad (61)$$

where $p_{t+1}^n(\varphi, F', k')$ denotes the probability of repaying at $t + 1$, in which $k' = k_t^*(\varphi, \tilde{x})$ and $F' = F_t^*(\varphi, \tilde{x})$.

2. The value of firm entry is given by

$$V_t^e(\varphi, s) = V_t(\varphi, sB_0).$$

$k_t^*(\varphi, sB_0)$, $F_t^*(\varphi, sB_0)$, and $B_t^*(\varphi, sB_0, sB_0)$ solve the optimal k' and F' respectively for entrants.

3. The measure of entrants is given by

$$\zeta_t(\varphi, s) = M_t \int \mathbb{1}\{V_t^e(\varphi, s) \geq 0\} d\omega(\varphi),$$

where $\omega(\varphi)$ denotes the distribution of initial φ .

4. The measure of surviving firms is given by

$$\begin{aligned}\eta_{t+1}(\varphi', \tilde{x}', B') &= \int \mathbb{1}\{\tilde{x}' \geq \underline{x}_t(\varphi')\} \times \mathbb{1}\{B' = B_t^*(\varphi, \tilde{x}, B)\} \\ &\quad \times \mathbb{1}\left\{\tilde{x}' = A' \varphi' (k_t^*(\varphi, \tilde{x}))^\alpha - c^f + (1 - \delta) k_t^*(\varphi, \tilde{x}) - F_t^*(\varphi, \tilde{x})\right\} \times j(\varphi'|\varphi) d\eta_t(\varphi, \tilde{x}, B) \\ &+ \int \mathbb{1}\{\tilde{x}' \geq \underline{x}_t(\varphi')\} \times \mathbb{1}\{B' = B_t^*(\varphi, sB_0, sB_0)\} \\ &\quad \times \mathbb{1}\left\{\tilde{x}' = A' \varphi' (k_t^*(\varphi, sB_0))^\alpha - c^f + (1 - \delta) k_t^*(\varphi, sB_0) - F_t^*(\varphi, sB_0)\right\} \times j(\varphi'|\varphi) d\zeta_t(\varphi, s)\end{aligned}$$

where $j(\varphi'|\varphi)$ denotes the density of the transition probability $J(\varphi'|\varphi)$, $\underline{x}(\varphi)$ denotes the cutoff adjusted net worth of defaulting. Note that A' is known at t as there exists no aggregate uncertainty.

5. The investment return is equal to

$$R_t \geq \frac{1}{\beta}.$$

6. The debt price schedule is characterized by

$$q_t^F(\varphi, F', k') = \frac{1}{R_{t+1}} \left[(p_{t+1}^n(\varphi, F', k') F' + (1 - p_{t+1}^n(\varphi, F', k')) \gamma_{t+1} k') \right].$$

3.3 Debt Decomposition

In Section 3.3 we assume that only unsecured debt can be rolled over indefinitely. This assumption is for clarification purposes but not crucial for our results. We now demonstrate that the analysis in 3.5 remains robust in a more generic scenario: firms can indefinitely roll over secured debts.

Without loss of generality, suppose firms issue two debts, Debt X and Debt Y, which are priced by

$$q_{jt}^X b_{jt+1}^X = \frac{1}{R_{t+1}} \left[(p_{jt+1}^n b_{jt+1}^X + (1 - p_{jt+1}^n) \gamma_{t+1}^X k_{jt+1}) \right], \quad (62)$$

$$q_{jt}^Y b_{jt+1}^Y = \frac{1}{R_{t+1}} \left[(p_{jt+1}^n b_{jt+1}^Y + (1 - p_{jt+1}^n) \gamma_{t+1}^Y k_{jt+1}) \right]. \quad (63)$$

b^X and b^Y denote the face value, and q_{jt}^X and q_{jt}^Y denote the price of Debt X and Y, respectively. If firm j defaults at $t + 1$, the creditors of Debt X retrieve collateral $\gamma_{t+1}^X k_{jt+1}$, and the creditors of Debt Y retrieve collateral $\gamma_{t+1}^Y k_{jt+1}$. $\gamma_t^X + \gamma_t^Y = \gamma_t$ for all t .

In addition, we assume that Debt X is subject to no-Ponzi-game condition, whereas

the existing amount of Debt Y can be persistently rolled over such that

$$q_{jt}^Y b_{jt+1}^Y \leq b_{jt}^Y \quad (64)$$

Similar to Proposition 2, it is always optimal for firms to fully roll over the Debt Y, i.e., $q_{jt}^Y b_{jt+1}^Y = b_{jt}^Y$, instead of repaying the debt with firms' surplus.⁴ Therefore we can rewrite (63) into

$$q_{jt}^Y b_{jt+1}^Y = \frac{1}{R_{t+1}} E_t \left(f_{jt+1}^Y + q_{jt+1}^Y b_{jt+2}^Y \right), \quad (65)$$

where

$$f_{jt}^Y \equiv \begin{cases} 0, & \text{if repay} \\ \gamma_t^Y k_{jt}, & \text{if default} \end{cases}.$$

Combining (17) with (65), we obtain

$$q_{jt}^X b_{jt+1}^X = \frac{1}{R_{t+1}} E_t \left(f_{jt+1}^X + q_{jt+1}^X b_{jt+2}^X \right), \quad (66)$$

where

$$f_{jt}^X \equiv \begin{cases} \pi_{jt} - \text{div}_{jt} - I_{jt}, & \text{if repay} \\ \gamma_t^X k_{jt}, & \text{if default} \end{cases}.$$

f_{jt}^X and f_{jt}^Y can be viewed as the net payment from the borrower to the creditors of Debt X and Y, respectively.

The total income from issuing debts, $q_{jt}^X b_{jt+1}^X + q_{jt}^Y b_{jt+1}^Y$, can be decomposed into two components, $\tilde{F}_{jt}$ and $\tilde{B}_{jt}$

$$\tilde{F}_{jt} \equiv q_{jt}^X b_{jt+1}^X + \sum_{\tau=t}^{\infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t f_{j\tau+1}^Y = \sum_{\tau=t}^{\infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t f_{j\tau+1}^Y. \quad (67)$$

$$\tilde{B}_{jt} \equiv q_{jt}^Y b_{jt+1}^Y - \sum_{\tau=t}^{\infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t f_{j\tau+1}^Y = \lim_{\tau \rightarrow \infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t q_{j\tau}^Y b_{j\tau+1}^Y. \quad (68)$$

(67) is derived by imposing the no-Ponzi-game condition $\lim_{\tau \rightarrow \infty} \left(\Pi_{i=t}^{\tau} \frac{1}{R_{i+1}} \right) E_t q_{j\tau}^X b_{j\tau+1}^X =$

⁴The proof in Appendix 3.1 applies here: one need only to replace F in the proof with Debt X, and replace B in the proof with Debt Y.

0. It is immediate that $\tilde{F}_{jt} = q_{jt}^F F_{jt+1}$ and $\tilde{B}_{jt} = q_{jt}^B B_{jt+1}$ for all t . Note that

$$\tilde{B}_{jt} = \frac{1}{R_{t+1}} E_t \lim_{\tau \rightarrow \infty} \left(\prod_{i=t+1}^{\tau} \frac{1}{R_{i+1}} \right) E_t q_{j\tau}^B b_{j\tau+1}^B = \frac{1}{R_{t+1}} E_t \tilde{B}_{jt+1}. \quad (69)$$

The bubble component is expected to be perfectly rolled over. $b_{jt}^X + b_{jt}^Y$ can be accordingly decomposed into F_{jt} and B_{jt} . Since B_{jt} gets rolled over, the optimization problem is indeed equivalent to choosing optimal investment and the external finance through fundamental component that equals the NPV of net flows to creditors. In a nutshell, the optimization problem is mathematically equivalent to the one described by (38) and (39).

It is straightforward to show that the decomposition above pertains to firms issuing any $n \in N^+$ types of debts, regardless of whether the debts are secured. Provided that a firm can indefinitely roll over a fraction of its debts, it solves the optimization problem as if it issues only the two types of debts, F and B , which are defined in Section 3.3.

Hence we can always decompose the total income from issuing debts into a fundamental component, which is equal to the net present value of the net flows to creditors, and an indefinite rollover component, which is backed by the expectation of future rollover. More importantly, we can treat the two components equivalently as the two debts F and B that are introduced in Section 3.3.

3.4 The Sustainability of Bubbles

The aggregate credit bubble B_A can be decomposed into

$$B_A = B_I + B_N,$$

where B_I represents the aggregate credit bubble of incumbent firms, B_N represents the aggregate credit bubble of entrants. According to Equation (22) and (31), B_I follows

$$\begin{aligned} B_I' &= \int \left[\left(q_*^B(\varphi, \tilde{x}) \right)^{-1} B \right] \cdot [p_*^n(\varphi, \tilde{x})] d\eta(\varphi, \tilde{x}, B) \\ &= \int \left[\left(\frac{1}{R} \cdot p_*^n(\varphi, \tilde{x}) \right)^{-1} B \right] \cdot [p_*^n(\varphi, \tilde{x})] d\eta(\varphi, \tilde{x}, B) \\ &= R \int B \cdot d\eta(\varphi, B, k') = R B_A, \end{aligned} \quad (70)$$

where $\eta(\varphi, \tilde{x}, B)$ denotes the measure of the incumbents and entrants,⁵ $q_*^B(\varphi, \tilde{x})$ and $p_*^n(\varphi, \tilde{x})$ respectively denote the price of bubble component and the probability of repayment along a BGP, when firms choose the optimal k' and F' given state variables φ and $\tilde{x}$. The law of motion for the B_A is described by

$$B'_A = RB_A + p_B B_0 N'. \quad (71)$$

Here, $B_N = p_B B_0 N$, where N denotes the number of entrants. The ratio $\frac{B'_A}{Y'}$ follows

$$\frac{B'_A}{Y'} = R \frac{B_A}{Y} \frac{Y}{Y'} + \frac{p_B B_0 N'}{Y'} = \frac{R}{1+g} \frac{B_A}{Y} + \frac{p_B B_0 N_0}{Y_0}, \quad (72)$$

where $\frac{N_0}{Y_0}$ denotes the number of entrants normalized by the aggregate output along a BGP. Equation (72) uses the fact that along a BGP, the number of entrants, the aggregate output Y , and the aggregate credit bubble B_A all increase at the rate g . Insofar as $\frac{R}{1+g} < 1$, i.e., $\beta(1+g) > 1$, Equation (72) implies that along a BGP, $\frac{B'_A}{Y'}$ stays at the fixed point $\left(1 - \frac{R}{1+g}\right)^{-1} \frac{p_B B_0 N_0}{Y_0}$.

3.5 An Example

This appendix uses a stylized example to help develop the intuition of credit bubbles and understand the assumptions in our full model.

3.5.1 Credit Bubbles

Consider a small open economy with a unit continuum of risk-neutral entrepreneurs, where each entrepreneur operates a firm. For simplicity, we assume that every firm is a shell company that generates zero operating profit ($\pi = 0$). Entrepreneurs have the option to save in a competitive and frictionless international financial market, which offers a risk-free interest rate R on deposits.

The only uncertainty in the economy is an idiosyncratic death shock that occurs iid across firms and time, with prob $1 - p$. The death shocks are realized at the beginning of each period: if it draws a death shock, a firm exits the market immediately and permanently before debt repayment is made. The exiters are replaced by the *same* amount

⁵Here entrants can be viewed as incumbents with $\tilde{x} = sB_0$ and $B = sB_0$, where $s = 1$ with prob p_B , and $s = 0$ with prob $1 - p_B$.

of new entrants. Firms can raise funds from the international financial market. Debts are unbacked by physical collateral. For simplicity, we assume there exists *no voluntary default*—an assumption we relax later. To compensate for the (involuntary) default risk, a firm borrows from the financial market by paying a *borrowing rate* $\frac{R}{p}$.

The intertemporal budget constraint of firm j can be written as

$$D_{jt} = \frac{R}{p} D_{jt-1} + div_{jt}. \quad (73)$$

D_{jt} denote the amount firm j borrows at t . $D_{jt-1} = 0$ for new firms. Solving (73) forward leads to

$$D_{jt} = -\sum_{\tau=1}^{\infty} \left(\frac{p}{R}\right)^{\tau} div_{jt+\tau} + \lim_{\tau \rightarrow \infty} \left(\frac{p}{R}\right)^{\tau} D_{jt+\tau}. \quad (74)$$

Here the net flow to lenders (conditional on repayment) at t is equal to $-div_t$.

Similar to (3), the second term on the right-hand-side represents the difference between the value of debt and the NPV of the expected net flows to the financial market. We denote the difference *credit bubbles*.

3.5.2 Indefinite Rollover

Next we show that credit bubbles can be replicated with a certain type of debts. Assume that firms are not binded by no-Ponzi-game condition. Specifically, firms issue two types of debts, D^F and D^B . Similar to the standard debt in the literature, D^F is subject to no-Ponzi-game condition, $\lim_{\tau \rightarrow \infty} \left(\frac{p}{R}\right)^{\tau} D_{jt+\tau}^F = 0$, i.e., D^F cannot be rolled over indefinitely.

Moreover, we follow [Hellwig and Lorenzoni \(2009\)](#) by assuming that, D^B can be rolled over permanently, but is subject to the following constraints:

$$D_{jt}^B \leq \begin{cases} \frac{R}{p} D_{jt-1}^B, & \text{(incumbent firms)} \\ B_0, & \text{(new firms)} \end{cases}. \quad (75)$$

To put it differently: new firms can borrow D^B up to B_0 ; afterwards firms can roll over the initial stock B_0 but cannot issue new D^B . This debt is analogous to the "unbacked public debt" by [Hellwig and Lorenzoni \(2009\)](#).

What if firms are not subject to the borrowing limits in (75), and instead, can borrow unconstrained level of D^B ? Suppose at t firm j borrows $D_{jt}^B = M + \frac{R}{p} D_{jt-1}^B$ and rolls it over from period to period, i.e., $D_{j\tau}^B = \frac{R}{p} D_{j\tau-1}^B \forall \tau > t$. Under such a circumstance, one can

easily derive $D_{jt}^F = -\sum_{\tau=1}^{\infty} \left(\frac{p}{R}\right)^{\tau} \text{div}_{jt+\tau}$ using (74). The corresponding NPV of dividends is

$$\sum_{\tau=0}^{\infty} \left(\frac{p}{R}\right)^{\tau} \text{div}_{jt+\tau} = M + D_{jt}^F + \sum_{\tau=1}^{\infty} \left(\frac{p}{R}\right)^{\tau} \text{div}_{jt+\tau} = M.$$

Hence, in the absence of borrowing limits, a firm can choose $M \rightarrow +\infty$ to have an infinite NPV of expected dividends.

We have the following Proposition.

Proposition 2. $\forall t$, it is optimal for a firm j to choose

$$D_{jt}^B = \begin{cases} \frac{R}{p} D_{jt-1}^B, & (\text{incumbent firms}) \\ B_0, & (\text{new firms}) \end{cases}. \quad (76)$$

Proof. At t , the NPV of expected dividends is equal to

$$\begin{aligned} \sum_{\tau=t}^{\infty} \left(\frac{p}{R}\right)^{\tau-t} \text{div}_{j\tau} &= \sum_{\tau=t}^{\infty} \left(\frac{p}{R}\right)^{\tau-t} \left(D_{j\tau} - \frac{R}{p} D_{j\tau-1} \right) \\ &= \sum_{\tau=t}^{\infty} \left(\frac{p}{R}\right)^{\tau-t} \left[D_{j\tau}^F + D_{j\tau}^B - \frac{R}{p} (D_{j\tau-1}^F + D_{j\tau-1}^B) \right] \\ &= \lim_{\tau \rightarrow \infty} \left(\frac{p}{R}\right)^{\tau-t} \left(D_{j\tau}^F + D_{j\tau}^B \right) - \frac{R}{p} D_{jt-1} \\ &= \lim_{\tau \rightarrow \infty} \left(\frac{p}{R}\right)^{\tau-t} D_{j\tau}^B - \frac{R}{p} D_{jt-1}. \end{aligned} \quad (77)$$

Given constraint (75), firm j maximizes its NPV of expected dividends by maximizing the path $\left\{ D_{j\tau}^B \right\}_{\tau=t}^{\infty}$:

$$D_{jt}^B = \begin{cases} \frac{R}{p} D_{jt-1}^B, & (\text{incumbent firms}) \\ B_0, & (\text{new firms}) \end{cases}.$$

□

One can also use the reasoning above to prove that if a firm is allowed to roll over $D_{j\tau}^B$ perpetually starting from $\tau = t + 1$, the firm would always maximize D_{jt}^B .

It is immediate from (76) that $D_{jt}^F = -\sum_{\tau=1}^{\infty} \left(\frac{p}{R}\right)^{\tau} \text{div}_{jt+\tau}$ $D_{jt}^B = \lim_{\tau \rightarrow \infty} \left(\frac{p}{R}\right)^{\tau} D_{jt+\tau}$, i.e., D_{jt}^B is equal to the credit bubble component.

Default, Lenders, and Self-enforcing Debt Limit So far we have demonstrated that the debt D^B is equal to the credit bubble of a borrower. We also show that borrowers are willing to borrow as much D^B as they can. As to lenders, obviously, in a world without voluntary default, lenders are indifferent between these debts as they yield same expected return.

Next we illustrate why rational lenders are willing to lend this type of debts, in a scenario with potential *voluntary default*.

Assume that firms can voluntarily default on their debt: if they default, they exit the market and lose the access to the credit market. Moreover, we assume that firms cannot issue negative dividends. (76) implies that D_{jt}^B is perpetually repaid by future debts (from lenders): $\forall t, j$

$$D_{jt}^B = \frac{p}{R} D_{jt+1}^B. \quad (78)$$

(78) ensures that firms have no incentive to *voluntarily* default on their debt, i.e., D_{jt}^B is self-enforcing.⁶ Therefore the right-hand-side of (78) can be interpreted as *collateral*, despite that the collateral is future debts instead of assets (or future income) in a typical collateral constraint. (78) can be viewed that, *given* future debt $\frac{p}{R} D_{jt+1}^B$, lenders provide the maximum amount of D_{jt}^B that is incentive compatible with repayment.

Unlike D^B , D^F is unbacked by any physical collateral or indefinite rollover, therefore the upper limit of D^F is

$$D_{jt}^F \leq 0. \quad (79)$$

Specifically, defaulting at $t + 1$ allows firm j to keep $\frac{R}{p} (D_{jt}^F + D_{jt}^B)$, whereas the loss is

$$D_{jt+1}^F + D_{jt+1}^B + \sum_{\tau=2}^{\infty} \left(\frac{p}{R}\right)^{\tau} div_{jt+\tau} = D_{jt+1}^B.$$

Hence the incentive compatible constraint is $\frac{R}{p} (D_{jt}^F + D_{jt}^B) \leq D_{jt+1}^B$, i.e., (79).⁷

⁶ D_{jt}^B here is analogous to "self-enforcing debts" and "unbacked public debts" by Hellwig and Lorenzoni (2009) and "bubbly collateral" by Martin and Ventura (2016).

⁷We use the result that, if voluntary default does not occur (i.e., credit contracts are incentive compatible), $D_{jt}^F = -\sum_{\tau=1}^{\infty} \left(\frac{p}{R}\right)^{\tau} div_{jt+\tau}$ and $D_{jt}^B = \frac{p}{R} D_{jt+1}^B$.

Firm Entry, Exit, and Sustainability of Bubbles It is easy to derive the law of motion for the aggregate credit bubble B^A

$$\begin{aligned} B_{t+1}^A &= \frac{R}{p} \cdot p \cdot B_t^A + p \cdot B_0 \\ &= R \cdot B_t^A + p \cdot B_0. \end{aligned} \quad (80)$$

In the absence of aggregate uncertainty, the aggregate credit bubble converges to $B^A = \frac{pB_0}{1-R}$. It is obvious that the law of motion is stationary only if $R < 1$; otherwise the size of credit bubbles would eventually outgrow and deplete the resources in the world.

Firm entry and exit are crucial for bubbles to be sustainable, if firms earn positive operation profit and can issue positive dividends in the long run, as in our full model. In the absence of firm entry and exit, i.e., $p = 1$, the value of a firm (the NPV of dividends) become infinity as $R < 1$. For the value of firms to be finite, one need $p < R < 1$.

In our full model, the long run growth rate is equal to the growth rate of firm numbers, g . Bubbles are sustainable along a BGP if $R < 1 + g$.

An Unexpected Increase in Credit Bubbles Finally, consider an unexpected MIT shock (with zero probability) at t : incumbent firms are allowed to borrow $D_t^B = \frac{R}{p}D_{jt-1}^B + \Delta$. The shock is driven by a market sentiment change and is captured by Δ : firms are now allowed to borrow Δ in addition to $\frac{R}{p}D_{jt-1}^B$, the required amount for rollover.

Furthermore, we assume that, for an unspecified reason, firms pay zero dividends. This assumption makes firms analogous to the financially constrained firms in our full model. As to an incumbent firm, according to budget constraint (74), we get

$$D_{jt}^F = \frac{R}{p}D_{jt-1}^F - \Delta. \quad (81)$$

(81) implies that $\frac{\partial D_{jt}^F}{\partial \Delta} < 0$. Since firms cannot pay dividends, they use the newly issued D_{jt}^B to reduce D_{jt}^F .

Note that in our full model, financially constrained firms also pay *zero* dividends. The major difference in the full model is that firms choose not only future repayment F' , but also future capital stock k' . In Section 4.1 we study the impulse responses following an unexpected increase in credit bubbles. Firms can either save the increase in F' (reduce F') or save in their capital stock k' .

4 Numerical Appendix

Before we proceed to the algorithm, it is useful to establish some results regarding the optimal behavior of firms. The optimization problem defined in (38) and (39) is well investigated by the literature.⁸ 1) there exist a cutoff level of adjusted net worth, $\underline{x}(\varphi)$, below which firms default; 2) there exist a cutoff level of adjusted net worth, $\bar{x}(\varphi)$, above which firms become financially unconstrained, i.e., firms can perpetually choose the unconstrained level of capital; 3) firms with $\tilde{x} \in [\underline{x}(\varphi), \bar{x}(\varphi)]$ are financially constrained and set their dividends to zero.

Balanced Growth Path Along a BGP, state space can be fully summarized by idiosyncratic productivity φ , and adjusted net worth $\tilde{x}$, which is defined by (37). We follow the standard numerical approach to solve the problem.

1. Solve for $p^n(\varphi, k', F')$, the probability of repaying in the subsequent period, $q^F(\varphi, k', F')$, debt price schedule, and $\underline{x}(\varphi)$, the cutoff level of adjusted net worth $\tilde{x}$ for firms to default. Make an initial guesses for function $p^n(\varphi, k', F')$. Note that B' does not affect the probability of repaying, since B can always be rolled over and does not affect the adjusted net worth in the subsequent period. Then characterize the corresponding debt price schedule $q^F(\varphi, k', F')$ from Equation (20). Next calculate the maximum amount of $q^F(\varphi, k', F')F' - k'$ for various k' and F' . Denote the amount by $\underline{M}(\varphi)$. Since dividends are equal to $\tilde{x} + q^F(\varphi, k', F')F' - k'$, negative dividends are inevitable if adjusted net worth $\tilde{x} + \underline{M}(\varphi) < 0$. Therefore the cutoff level $\underline{x}(\varphi)$ is equal to $-\underline{M}(\varphi)$. Given idiosyncratic productivity level φ , if the adjusted net worth $\tilde{x} < \underline{x}(\varphi)$, the dividend of the firm is inevitably negative, and the firm has to default. Provided $\underline{x}(\varphi)$, we can then update the guess of probability $p^n(\varphi, k', F')$. Iterate this process until both $p^n(\varphi, k', F')$ and $\underline{x}(\varphi)$ converge.

2. Solve for $v_c^u(\varphi)$, the continuation value function of financially unconstrained firms, $\bar{x}(\varphi)$, the cutoff level of adjusted net worth $\tilde{x}$ for firms to be financially unconstrained, and $k^u(\varphi)$ and $F^u(\varphi)$, the policy functions for financially unconstrained firms. $k^u(\varphi)$ can be solved by equating the marginal return on investment to risk-free interest rate. Unconstrained firms are indeed indifferent between $-F'$ (saving) and paying dividends

⁸See the discussions of Ottonello and Winberry (2020), Khan and Thomas (2013), and Arellano et al. (2019) study a more complicated version of this problem, with aggregate states included in the state space. Ottonello and Winberry (2020) and Jeenas (2019) solve the problem at steady states. Like us, they can remove the aggregate states from the state space.

as they remain unconstrained in the subsequent periods with probability one. To resolve the indeterminacy, we follow [Khan and Thomas \(2013\)](#) by assigning the minimum saving policy to $F^u(\varphi)$ such that unconstrained firms borrow maximum amount of $F^u(\varphi)$ (or equivalently, save the minimum amount) that ensures them to maintain unconstrained in the future, regardless of the realization of states. $F^u(\varphi)$ follows

$$F^u(\varphi) = \min_{\varphi'} \left\{ \hat{x}(\varphi', \varphi) - k^u(\varphi') + \frac{1}{R} F^u(\varphi') \right\}, \quad (82)$$

where $\hat{x}(\varphi', \varphi) \equiv A\varphi'(k^u(\varphi))^\alpha - c^f + (1 + \delta)k^u(\varphi)$. (82) implies that, by promising to repay $F^u(\varphi)$ in the subsequent period, firms can raise unconstrained level of capital $k^u(\varphi)$ without any default risk in the subsequent period, regardless of the realization of φ' . The debt price is equal to $\frac{1}{R}$ because of zero default risk. If firms borrow more than $F^u(\varphi)$, they can not guarantee the availability of unconstrained policy functions $k^u(\varphi)$ and $F^u(\varphi)$ in subsequent periods. $F^u(\varphi)$ can be solved by iterative guessing and verifying. Firms can implement the policy functions without negative dividends if adjusted net worth $\tilde{x} - [k^u(\varphi) - \frac{1}{R}F^u(\varphi)] \geq 0$. Hence the cutoff level $\bar{x}(\varphi)$ is equal to $k^u(\varphi) - \frac{1}{R}F^u(\varphi)$. Given idiosyncratic productivity level φ , if the adjusted net worth $\tilde{x} \geq \bar{x}(\varphi)$, firms can not only choose unconstrained level of k' and F' , but also remain unconstrained permanently by persistently following the unconstrained policy functions. The continuation value $v_c^u(\varphi)$ follows

$$v_c^u(\varphi) = -k^u(\varphi) + \frac{1}{R}F^u(\varphi) + \frac{1}{R} \sum_{i=1}^{N_\varphi} \omega(\varphi_i|\varphi) [\hat{x}(\varphi_i, \varphi) + v_c^u(\varphi_i)],$$

where $\omega(\varphi_i|\varphi)$ denotes the transition probability from φ to φ_i , N_φ represents the number of grids for φ . Note that in this appendix we denote the transition probability by discrete distribution since functions are defined on discrete grids in the algorithm. We use Tauchen method to approximate the AR1 process (13) to get the transition probability $\omega(\varphi_i|\varphi)$. $v_c^u(\varphi)$ can be solved by iterative guessing and verifying.

3. Solve for $V(\varphi, \tilde{x})$, the value function of incumbent firms, $v^c(\varphi, \tilde{x})$, the value function of financially constrained firms, and $k^c(\varphi, \tilde{x})$ and $F^c(\varphi, \tilde{x})$, the policy functions for financially constrained firms. Constrained firms issue zero dividend since the return from self-financing the firms is higher than risk-free rate, which is also the investment return the shareholders can get on dividends. Therefore we use bisection method to characterize $F(\varphi, \tilde{x}, k')$, the level of fundamental credit that satisfies zero-dividend given φ , $\tilde{x}$, k' . Substitute F' in debt price schedule $q^F(\varphi, k', F')$ with $F' = F(\varphi, \tilde{x}, k')$ to get $q_c^F(\varphi, \tilde{x}, k')$, the

debt price schedule for constrained firms given $\varphi, \tilde{x}, k'$. The value function of financially constrained firms $v^c(\varphi, \tilde{x})$ follows

$$v^c(\varphi, \tilde{x}) = \max_{k'} \left\{ \tilde{x} - k' + q_c^F(\varphi, \tilde{x}, k') F(\varphi, \tilde{x}, k') + \frac{1}{R} \sum_{i=1}^{N_\varphi} \omega(\varphi_i | \varphi) V(\varphi', \tilde{x}') \right\},$$

where $V(\varphi, \tilde{x})$ follows

$$V(\varphi, \tilde{x}) = \mathbb{1}\{\tilde{x} \geq \bar{x}(\varphi)\}(\tilde{x} + v_c^u(\varphi)) + \mathbb{1}\{\underline{x}(\varphi) \leq \tilde{x} < \bar{x}(\varphi)\} v^c(\varphi, \tilde{x}).$$

$v^c(\varphi, \tilde{x})$ and $V(\varphi, \tilde{x})$ can be solved by iterative guessing and verifying. Use grid search to pin down $k^c(\varphi, \tilde{x})$, the optimal k' for constrained firms. Substitute k' in $F(\varphi, \tilde{x}, k')$ with $k^c(\varphi, \tilde{x})$ to obtain $F^c(\varphi, \tilde{x})$.

4. Compute the distribution of firms. As for any firm, policy functions can be represented by

$$k^*(\varphi, \tilde{x}) = \mathbb{1}\{\tilde{x} \geq \bar{x}(\varphi)\} k^u(\varphi) + \mathbb{1}\{\underline{x}(\varphi) \leq \tilde{x} < \bar{x}(\varphi)\} k^c(\varphi, \tilde{x}),$$

$$F^*(\varphi, \tilde{x}) = \mathbb{1}\{\tilde{x} \geq \bar{x}(\varphi)\} F^u(\varphi) + \mathbb{1}\{\underline{x}(\varphi) \leq \tilde{x} < \bar{x}(\varphi)\} F^c(\varphi, \tilde{x}).$$

Substitute k' and F' in $p^n(\varphi, k', F')$ with policy functions $k^*(\varphi, \tilde{x})$ and $F^*(\varphi, \tilde{x})$ respectively to get $p^n(\varphi, \tilde{x})$, the probability of repaying conditional on φ and $\tilde{x}$.⁹ $p^n(\varphi, \tilde{x})$ is used to characterize the law of motion for B , according to (22) and (31). Now given the law of motion for B , policy functions $k^*(\varphi, \tilde{x})$ and $F^*(\varphi, \tilde{x})$, cutoff levels $\bar{x}(\varphi)$ and $\underline{x}(\varphi)$, and the transition matrix of φ , we construct the transition matrix Γ , of $\varphi, \tilde{x}, B$. Along BGP, the relative distribution of firms remain unchanged. The detrended stationary distribution vector $\tilde{\eta}_t$ follows the law of motion

$$\tilde{\eta}_{t+1} = \frac{1}{1+g} \Gamma \times \tilde{\eta}_t + \tilde{\zeta}, \quad (83)$$

where $\tilde{\zeta}$ denotes the detrended distribution vector of entrants. Iterate (83) until $\tilde{\eta}_t$ converges to get the stationary distribution.

Transition Dynamics We characterize the perfect foresight equilibrium paths along which:

i) unexpected shocks arrive at $t = 0$; ii) prior to $t = 0$ and after $t = 99$, the economy is along BGP. Use backward induction to compute: i) the value functions value functions $V_t(\varphi, \tilde{x})$

⁹ $p^n(\varphi, \tilde{x}) = 1$ if $\tilde{x} \geq \bar{x}(\varphi)$.

and $V_t^e(\varphi)$, ii) policy functions $k_t^*(\varphi, \tilde{x})$, $F_t^*(\varphi, \tilde{x})$ and $B_t^*(\varphi, \tilde{x}, B)$, iii) and cutoff levels $\bar{x}_t(\varphi)$, $\underline{x}_t(\varphi)$, from $t = 99$ to $t = 0$. Then construct transition matrix Γ_t and the detrended distribution vector of entrants $\tilde{\zeta}_t$ for every t from $t = 0$ to $t = 99$. Finally, iterate forward the law of motion

$$\tilde{\eta}_{t+1} = \frac{1}{1+g} \Gamma_t \times \tilde{\eta}_t + \tilde{\zeta}_{t+1}, \quad (84)$$

to get the detrended distribution vector of firms for every t from $t = 1$ to $t = 99$.